# On $L \log L$ estimates

by

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**Motivation: It is good to have $L^p$ or $L \log L$ proof without excursions in $L^1$ theory.**

P.Sjögren proved that non centered Gaussian Hardy-Littlewood maximal function is not weak type (1,1). L.Forzani, R.Scotto, P.Sjögren, and W.Urbina proved $L^p$ boundedness for $p > 1$.

L.Forzani& R.Scotto and independently J.García-Cuerva, G.Mauceri, P.Sjögren, and José-Luis Torrea proved that the higher-order Riesz transform for Gaussian measure associated with the Ornstein-Uhlenbeck differential operator $L = d^2/dx^2 - 2xd/dx$, $x \in \mathbf{R}$, need not be of weak type $(1,1)$.

S.Pérez proved that it is weak type $L(\log L)^{(k-2)/k}$ w.r.t. Gaussian measure for $k \geq 3$.

## Doob inequalities

Let $0 \leq x_n$ be a submartingale and

$$x_n^* = \sup_{0 \leq k \leq n} x_k.$$

A well-known Doob inequalities are

$$\|x_n^*\|_p \leq q\|x\|_p \quad 1 < p < \infty \quad 1/p + 1/q = 1 \tag{1}$$

and

$$\|x_n^*\|_1 \leq \frac{e}{e-1}(\|x_n \log^+ x_n\|_1 + 1). \tag{2}$$

They were proved by Doob. Later, Burkholder using new method, called by him *U,V special functions method*, find an alternative proof of (1).

Let me discuss Burkholder's method in much generality as I can do.

## Burkholder's U,V special functions method:

Let $\phi(x)$ and $\psi(x)$ be increasing positive functions, such that $0 \leq \phi(x) \leq \psi(x)$ for $x \geq a \geq 0$.

Set

$$v(x,y) \equiv x\phi(x) - \gamma y \psi(y)$$

and

$$u(x,y) \equiv \beta[x\phi(x) - y\psi(x)]$$

with some positive $\gamma \geq \beta$.

**Lemma**. *If*

$$v(x,y) \leq u(x,y) \qquad \text{for} \qquad a \leq y \leq x \qquad (3)$$

*and*

$$u(y,y) \leq u(x,y) \qquad \text{for} \qquad a \leq y,\ a \leq x \quad (4)$$

*then*

$$E(x_n^* \phi(x_n^*)) \leq \gamma E(x_n \psi(x_n))$$

**Proof:**

A crucial observation is that

$$x_{n+1}^* = x_n^* \vee x_{n+1} \tag{5}$$

Now, it is clear that

$$u(x \vee (y+d), y+d) = \begin{cases} u(x, y+d) \\ \text{or} \\ u(y+d, y+d) \end{cases}$$

In the second case (4) implies that $u(y+d, y+d) \le u(x, y+d)$. Hence,

$$u(x \vee (y+d), y+d) \le u(x, y+d) =$$

$$\beta[x\phi(x) - (y+d)\psi(x)] =$$

$$\beta[x\phi(x) - y\psi(x)] - \beta d\psi(x) = u(x,y) - \beta d\psi(x).$$

which together with (5) implies that

$$u(x_{n+1}^*, x_{n+1}) = u(x_n^* \vee (x_n + d_n), x_n + d_n) \le$$

$$u(x_n^*, x_n) - \beta d_n \psi(x_n^*).$$

Taking an expectation we get

$$Eu(x_{n+1}^*, x_{n+1}) \leq Eu(x_n^*, x_n) - \beta E(d_n \psi(x_n^*)).$$

But

$$E(d_n \psi(x_n^*)) =$$

$$E(E(d_n \psi(x_n^*) | x_n^*)) = E(\psi(x_n^*) E(d_n | x_n^*))$$

and $E(d_n | x_n^*) \geq 0$ by the definition of submartingale, then $E(d_n \psi(x_n^*)) \geq 0$ and thus using (3)

$$Ev(x_{n+1}^*, x_{n+1}) \leq Eu(x_{n+1}^*, x_{n+1}) \leq$$

$$Eu(x_n^*, x_n) \leq \cdots \leq Eu(x_0^*, x_0) = Eu(x_0, x_0) =$$

$$[E(x_0 \phi(x_0)) - E(x_0 \psi(x_0))] \leq 0,$$

or

$$Ev(x_n^*, x_n) = E(x_n^* \phi(x_n^*)) - \gamma E(x_n \psi(x_n)) \leq 0.$$

Lemma is proved.

**Necessary conditions for $\phi(x)$ and $\psi(x)$:**

Note, that (4) implies $u_1'(x,x) = 0$, hence

$$\beta[\phi(x) + x\phi'(x) - y\psi'(x)]\Big|_{x=y} = 0$$

or

$$\phi(x) + x\phi'(x) = x\psi'(x), \tag{6}$$

or

$$\psi(x) = \int_a^x \frac{\phi(t)}{t} dt + \phi(x). \tag{7}$$

**Corollary:** *Let $\phi(x)$ be an increasing positive function and $\psi(x)$ is given by (7). Let $u(x,y)$ and $v(x,y)$ are Burkholder's functions. Then (3) implies*

$$E(x_n^*\phi(x_n^*)) \leq \gamma E(x_n\psi(x_n))$$

**Conjecture:** *Let $\phi(x)$ be an increasing positive function and $\psi(x)$ is given by (7). Then (3) is valid too.*

**Example 1,** $1 < p < \infty$: Let $\phi(x) = x^{p-1}$, then it easy to see that $\psi(x) = qx^{p-1}$, where $q = p/(p-1)$ is the conjugant exponent.

**Example 2,** $p = 1$:

Let $x \geq 1$ $\phi(x) \equiv 1$, then $\psi(x) = \log x + 1$. In this case

$$v(x,y) = x - \gamma(y \log y + y)$$

$$u(x,y) = \beta[x - y \log x - y].$$

Now, we have to check only (3), i.e. to show that with some positive $\gamma$ and $\beta$ and for some $a > 0$

$$x - \gamma y \log y - \gamma y \leq \beta[x - y \log x - y] \qquad a \leq y \leq x.$$

This inequality is equivalent to

$$y[\beta \log x - \gamma \log y] + (\beta - \gamma)y \leq (\beta - 1)x$$

or

$$\beta y \log \frac{x}{y^{\gamma/\beta}} + (\beta - \gamma)y \leq (\beta - 1)x.$$

Since $\log x \leq x/e$ then the above inequality will follow from

$$\frac{\beta y x}{e y^{\gamma/\beta}} + (\beta - \gamma) y \leq (\beta - 1) x$$

or

$$y^{1-(\gamma/\beta)} + \frac{\beta - \gamma}{\beta} \frac{y}{x} \leq e \frac{\beta - 1}{\beta}.$$

Thus, the quantity

$$y^{1-(\gamma/\beta)} + (1 - (\gamma/\beta)) \frac{y}{x}$$

must be bounded uniformly in $x$ and $y$, which implies $\gamma = \beta$. That turns the above inequality in the following

$$1 \leq e \frac{\beta - 1}{\beta}.$$

which solution is $\beta \geq e/(e-1)$.

Since $\gamma = \beta$ and we are interesting in the smallest possible values for $\gamma$, then letting $\gamma = e/(e-1)$ we get (4).

Thus

$$v(x,y) = x - \frac{e}{e-1}\,(y \log y + y)$$

$$u(x,y) = \frac{e}{e-1}[x - y \log x - y].$$

and the application of the Lemma implies that

$$\|x_n^*\|_1 \le \frac{e}{e-1}\,(\|x_n \log(x_n)\|_1 + \|x_n\|_1)$$

provided that $x_n \ge 1$.

**Approach of Nazarov-Treil-Volberg:**

$$\mathcal{B}(f,F,L) = \sup_{0\le\varphi\in L^p_{\mathrm{loc}}(\mathbb{R})} \Big\{ \langle (M\varphi)^p\rangle_Q :$$

$$\langle\varphi\rangle_Q = f;\, \langle\varphi^p\rangle_Q = F;\, \sup_{R\supset Q}\langle\varphi\rangle_R = L \Big\}.$$

$$\mathcal{B}(f,F,L) \le pL^p - pqL^{p-1}f + q^pF$$

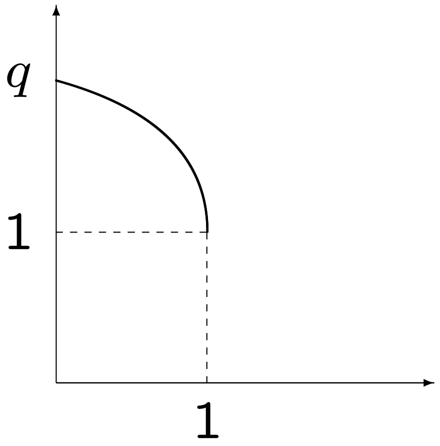
where $1/p + 1/q = 1$.

This finally gives us the inequality

$$\frac{1}{|J|}\int_J (Mf)^p \leq \frac{q^p}{|J|}\int_J f^p - \frac{1}{(p-1)}\left(\frac{1}{|J|}\int_J f\right)^p.$$

Melas:

$$\mathcal{B}(f,F,L) = \begin{cases} F\omega\left(\frac{pfL^{p-1}-(p-1)L^p}{F}\right)^p & L < qf \\ L^p + q^p(F - f^p) & L \geq qf \end{cases}$$



For the case $p = 1$ if one use

$$\alpha L - \beta f \log L + \gamma F$$

then it might be proved that

$$\frac{1}{|J|}\int_J Mf \le \frac{e}{e-1}\left(\frac{1}{|J|}\int_J f + \frac{1}{|J|}\int_J f\log f\right.$$

$$\left.-\left(\frac{1}{|J|}\int_J f\right)\log\left(\frac{1}{|J|}\int_J f\right)\right)$$

provided that $f \ge 1$ a.e. on $J$.

**Gross logarithmic Sobolev inequality:**

$$\int_{\mathbb{R}^d} |f(x)|^2 \log|f(x)|\gamma_d(dx) \le$$

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2\gamma_d(dx) + ||f||^2_{L^2(\gamma_d)}\log||f||_{L^2(\gamma_d)},$$

The products of $L$ and $\log L$ in the integrals on the right hand side of the inequality is so untypical that it might be a chance that this is not a coincidence.

# Thank you for your attention!