Product Rule and Chain Rule Estimates for Hajlasz Gradients on Doubling Metric Measure Spaces

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Abstract

In this paper we introduced Maximal Functions $N(f,x)$ of A.P. Calderón in the context of doubling metric measure spaces (X,d,μ). It is shown that these maximal functions are equivalent to the Hajlasz gradients. Using these maximal functions we prove L^s-norm estimates for the Product Rule and the Chain Rule for functions on (X,d,μ).

1 Introduction

We shall say that (X,d,μ) is a metric measure space if (X,d) is a metric space and μ is a Borel measure on X. Let $B_r(x)$ denote the ball of center x and radius $r>0$. If μ satisfies the condition

$$\mu(B_{2r}(x)) \leq C_D \mu(B_r(x))$$

with a constant C_D independent of x and r, then (X,d,μ) will be called a doubling metric measure space.

Following Hajlasz we shall say that a measurable function f has a gradient g in $L^p(X,d,\mu)$ if

$$|f(x)-f(y)| \leq d(x,y)(g(x)+g(y)) \tag{1.1}$$

holds for all x and $y \in X$ with $g \in L^p$. It can be shown that for $p>1$ there exists a unique $g=f^{\nabla}$ in L^p satisfying (1.1) and such that

$$\left\| f^{\nabla} \right\|_p = \inf_g \|g\|_p ,$$

where the infimum is taken over all g satisfying (1.1). We will call f^{∇} the Hajlasz gradient of f in L^p. Let f be a measurable function and $u \geq 1$, we define the Calderón Maximal Function $N_u(f)$ by

$$N_u(f,x) = \sup_{r>0} \frac{1}{r} \left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)-f(x)|^u \, d\mu(y) \right)^{1/u} .$$

We show in Theorem 1 the relationship between f^∇ and $N_u(f)$. In Theorem 2 we prove the main estimate for $N_u(F)$ that is needed to obtain the rules. We prove in Theorem 3 the Product Rule Estimate and in Theorem 4 the Chain Rule Estimate.

2 Theorems and Proofs

Theorem 1 *Let (X, d, μ) be a doubling metric measure space, f a measurable function, $1 < p < \infty$, and $1 \leq u < p$.*

a. *If f has the Hajlasz gradient in L^p, then there is a constant $C_1(u,p)$ such that*
$$\|N_u(f)\|_p \leq C_1(u,p) \left\|f^\nabla\right\|_p .$$

b. *If $N_u(f)$ is in L^p, the f has the Hajlasz gradient in L^p and there is a constant $C_2(u,p)$ such that*
$$\left\|f^\nabla\right\|_p \leq C_2(u,p) \|N_u(f)\|_p$$

Proof. We shall prove first part a). Since f^∇ satisfies (1.1), we have

$$\left(\frac{1}{\mu(B_r(x)}\int_{B_r(x)} |f(y)-f(x)|^u \, d\mu(y)\right)^{1/u} \leq$$

$$r\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} \left[f^\nabla(x)+f^\nabla(y)\right]^u d\mu(y)\right)^{1/u} \leq$$

$$r\left[f^\nabla(x)+\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} \left[f^\nabla(y)\right]^u d\mu(y)\right)^{1/u}\right].$$

Dividing both sides by r and taking supremum we get

$$N_u(f,x) \leq f^\nabla(x) + M_u(f^\nabla)(x)$$

where $M_u(f^\nabla)(x) = \sup\limits_r \left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} \left[f^\nabla(y)\right]^u d\mu(y)\right)^{1/u}$. Since $p > u$, $\left\|M_u(f^\nabla)\right\|_p \leq C(u,p) \left\|f^\nabla\right\|_p$, thus

$$\|N_u(f,x)\|_p \leq C_1(u,p) \left\|f^\nabla\right\|_p .$$

We shall prove now part b). Let x and y in X and $r = d(x,y)$. We have

$$|f(x)-f(y)| \leq \left(\frac{1}{\mu(B_r(x)}\int |f(x)-f(z)|^u d\mu(z)\right)^{1/u} +$$

$$\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(z)-f(y)|^u\,d\mu(z)\right)^{1/u}\leq$$

$$\frac{d(x,y)}{r}\left(\frac{1}{\mu(B_r(x)}\int_{B_r(x)}|f(x)-f(z)|^u\,d\mu(z)\right)^{1/u}+$$

$$\frac{d(x,y)}{r}\left[\frac{\mu(B_{2r}(y)}{\mu(B_r(x))}\right]^{1/u}\left(\frac{1}{\mu(B_{2r}(y))}\int_{B_{2r}(y)}|f(z)-f(y)|^u d\mu(z)\right)^{1/u}$$

$$\leq d(x,y)\left[C_D^{2/u}N_u(f,x)+C_D^{2/u}N_u(f,y)\right]. \qquad ((2.1))$$

Therefore, from the definition of f^{∇} it follows that

$$\left\|f^{\nabla}\right\|_p\leq C_D^{2/u}\left\|N_u(f)\right\|_p.$$

■

Theorem 2 *Let* $1\leq u<s\leq\infty$ *and* $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}$, $1<p\leq\infty,\ 1<q\leq\infty$. *Then*

$$\left\|\sup_{r>0}\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(y)-f(x)|^u\,|h(y)|^u\,d\mu(y)\right)^{1/u}\right\|_s\leq$$

$$C\left\|N_u(f)\right\|_p\left\|h\right\|_q,$$

where C *is a constant independent of* f *and* h.

Proof. We use inequality (2.1) to get

$$\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(y)-f(x)|^u\,|h(y)|^u\,d\mu(y)\right)^{1/u}\leq$$

$$\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}d^u(x,y)C_D^2\left[N_u(f,x)+N_u(f,y)\right]^u|h(y)|^u\,d\mu(y)\right)^{1/u}\leq$$

$$\left(\frac{C_D^2}{\mu(B_r(x)}\int_{B_r(x)}\left[N_u(f,x)+N_u(f,y)\right]^u|h(y)|^u d\mu(y)\right)^{1/u}\leq$$

$$N_u(f,x)C_D^{2/u}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|h(y)|^u\,d\mu(y)\right)^{1/u}+$$

$$C_D^{2/u}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}N_u^u(f,y)|h(y)|^u d\mu(y)\right)^{1/u}\leq$$

$$\leq C_D^{2/u}N_u(f,x)M_u(h)(x)+C_D^{2/u}M_u(N_u(f)\cdot h)(x).$$

Now, taking the supremum on the left hand side, then the $L^s - norm$ of both sides, and using that $1 \leq u < s \leq q$ we obtain

$$\left\| \sup_{r>0} \frac{1}{r} \left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|^u \, |h(y)|^u \, d\mu(y) \right)^{1/u} \right\|_s$$

$$\leq C_D^{2/u} \left[\|N_u(f) \cdot M_u(h)\|_s + \|M_u(N_u(f) \cdot h)\|_s \right]$$
$$\leq C_D^2 \left[\|N_u(f)\|_p \, \|M_u(h)\|_q + \|N_u(f) \cdot h\|_s \right]$$
$$\leq 2C_D^2 \, \|N_u(f)\|_p \, \|h\|_q \, .$$

■

Theorem 3 *(Product Rule Estimate)*
Let $1 < p_1, \; p_2 < \infty, 1 < q_1, q_2 \leq \infty, \; 1 < s \leq \infty, 1 < u < s$ *and* $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. *Then there is a constant* C *independent of* f *and* g *such that*

$$\|N_u(fg)\|_s \leq C \left[\|N_u(f)\|_{p_1} \, \|g\|_{q_1} + \|N_u(g)\|_{p_2} \, \|f\|_{q_2} \right].$$

Proof. We write

$$N_u(fg)(x) = \sup_r \frac{1}{r} \left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)g(y) - f(x)g(x)|^u \, d\mu(y) \right)^{1/u}$$

$$\leq \sup_r \frac{1}{r} \left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|^u \, |g(y)|^u d\mu(y) \right)^{1/u} +$$

$$\sup_r \frac{1}{r} \left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |g(y) - g(x)|^u \, |f(x)|^u \, d\mu(y) \right)^{1/u} \leq$$

$$\sup_r \frac{1}{r} \left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|^u \, |g(y)|^u \, d\mu(y) \right)^{1/u} + N_u(g,x)f(x).$$

We compute now the $L^s - norm$ of both sides, and use Theorem 2 and Hölder's inequality to estimate the terms on the right hand side respectively we obtain

$$\|N_u(fg)\|_s \leq C \left[\|N_u(f)\|_{p_1} \, \|g\|_{q_1} + \|N_u(f)\|_{p_2} \, \|g\|_{q_2} \right].$$

■

Theorem 4 *(Chain Rule Estimate)*
Let $F \in C^1(\mathbb{C}), H(z) = \sup_{|w|<|z|} |F'(w)|, \; \frac{1}{s} = \frac{1}{p} + \frac{1}{q}, \; 1 < s \leq \infty, \; 1 < p \leq \infty, \; 1 < q \leq \infty, \; 1 \leq u < s.$ *Then*

$$\|N_u(F \circ g)\|_s \leq C \, \|H \circ g\|_p \, \|N_u(g)\|_q$$

with C *independent of* F *and* g.

Proof. Observe that $\sup_{0\leq\lambda\leq 1} |F'(\lambda z_1 + (1-\lambda)z_2)| \leq H(z_1)+H(z_2)$ for any z_1, z_2. Then applying the Mean Value Theorem we have,

$$|F(g(y)) - F(g(x))| \leq \sup_{0\leq\lambda\leq 1} |F'(\lambda g(y) + (1-\lambda)g(x))|\,|g(y)-g(x)|$$

$$\leq [H(g(y)) + H(g(x))]\,|g(y)-g(x)|.$$

Therefore

$$N_u(F\circ g, x) = \sup_{r>0} \frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int |F(g(y)) - F(g(x)|^u\,d\mu(y)\right)^{1/u}$$

$$\leq \sup_r \frac{1}{r}\left(\frac{1}{\mu(B_r(x)}\int_{B_r(x)} |g(y)-g(x)|^u\,H^u(g(y))d\mu(y)\right)^{1/u} + H(g(x))N_u(g,x).$$

We compute now the L^s-norm of both sides, and using Theorem 2 and Hölder's inequality to estimate the terms on the right hand side respectively, we obtain

$$\|N_u(F\circ g)\|_s \leq C\,\|H\circ g\|_p\,\|N_u(g)\|_q$$

which concludes the proof of Theorem 4. ■

Note: The explicit formula for H in Theorem 4 was suggested by Michael Christ. Personal Communication.

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