Hereditary and Path Coalgebras

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Introduction

It is a well-known that, over an algebraically closed field and up to Morita equivalence, finite dimensional hereditary algebras are exactly the finite dimensional path algebras (see e.g. [Be]). In this note, we give short proof of the dual of this fundamental fact, which holds for arbitrary coalgebras and path coalgebras of quivers, without finiteness assumptions.

The results of this article were recently obtained differently in [JMLS] where hereditary coalgebras are studied via a notion of formal smoothness. Here we present a short and direct approach.

Background material may be found in [C,Mo,Sw]. We let Q be a quiver and let kQ denote the path coalgebra (see [CMo] or [C]) over the field k, with comultiplication Δ. For a path p, $s(p)$ denotes the starting vertex. Let $\{C_n\}$ denote the coradical filtration of the coalgebra C. By the quiver of C, we mean the Ext-quiver as in [C,CMo]. A coalgebra is said to be *hereditary* [NTZ] if homomorphic images of injective comodules are injective.

Our main result is

Theorem 1 *Every pointed hereditary coalgebra is isomorphic to the path coalgebra of its quiver.*

Corollary 2 *Every hereditary coalgebra over an algebraically closed field is Morita-Takeuchi equivalent to the path coalgebra of its (Ext-) quiver.*

To prove these statements we rely on

Lemma 3 *Let C be a coalgebra with a hereditary subcoalgebra D. If $C_1 \subset D$, then $C = D$.*

Proof. Let C and D be as in the hypothesis. First notice that if $D \wedge C_0 = D$, then $D = C$ (dual Nakayama Lemma [Sw]). Consequently (since $D \wedge C_0$ is a subcoalgebra containing D) we may assume that $D \wedge C_0 = C$; i.e.,

$$\Delta C \subset D \otimes C + D \otimes C_0$$

Therefore C/C_0 is a left D - comodule. The inclusion $D/D_0 \hookrightarrow C/C_0$ splits since D/D_0 is injective. Contensoring the inclusion with C_0 (i.e. taking socles) yields the injection $D_1/D_0 \hookrightarrow C_1/C_0$. But $C_1/C_0 = D_1/D_0$ is the socle of C/C_0. This forces $(C/D)_0 = 0$ and hence $C/D = 0$. This completes the proof of the Lemma. ■

Proof of Theorem and Corollary: Every coalgebra is Morita-Takeuchi equivalent to a basic coalgebra. Over an algebraically closed field basic coalgebras are precisely the pointed ones. Now let D be a hereditary pointed coalgebra. Then by [CMo], D embeds in the path coalgebra C of its quiver so that the copy of D contains C_1. By the Lemma, the image of D is all of C. ■

Conversely we have:

Theorem 4 *Every path coalgebra is hereditary.*

Proof. Let $C = kQ$. By [NTZ, Theorem 4] it suffices to show that C/S is injective for simple right coideals $S = kg$, where g is a vertex. Thus, letting I denote an indecomposable injective right coideal containing S, it suffices to show that I/S is injective. Here of course I is an injective hull of S, and I is isomorphic to the span of paths ending at g.

Let A_g denote the set of arrows ending at g. For each $\alpha \in A_g$, let $I(\alpha)$ be an injective hull of $ks(\alpha)$.

For all $\alpha \in A_g$,we define a map $f_\alpha : I \to I(\alpha)$ by letting $d_\alpha \in Hom_k(C, k)$ be defined by $d_\alpha(p) = \delta_{a,p}$, for paths p (Kronecker δ), and setting $f_\alpha = \leftharpoonup d_\alpha = (d_\alpha \otimes 1)\Delta$ using the "hit" action, see [Mo]. The map f_α is a right comodule homomorphism. The reader can quickly check that f_α can be seen to truncate paths ending at with α, i.e., $f_\alpha(\alpha p) = p$, $f_\alpha(\alpha) = s(\alpha)$ and is zero on paths not ending with α.

Now set

$$f = \oplus_{\alpha \in A_g} f_\alpha : I \to \oplus_{\alpha \in A_g} I(\alpha).$$

Then it is easy to see that f is onto, and that the kernel of f is kg. The image is an injective comodule since the direct sum of injective comodules is injective. This shows that is I/S injective as required. ■

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