A BRIEF INTRODUCTION TO COALGEBRA REPRESENTATION THEORY

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Abstract

In this survey, we review fundamental properties of coalgebras and their representation theory. Following J.A. Green we present the block theory of coalgebras using indecomposable injectives comodules. Using the cohom and cotensor functors we state Takeuchi-Morita equivalence and use it to sketch the proof of existence of "basic" coalgebras, due to the author and S. Montgomery. This leads to a discussion of theory of path coalgebras, quivers and representations. Some quantum and algebraic group examples are given.

1 Introduction

This survey article is aimed at algebraists who are not neccesarily specialists in coalgebras and Hopf algebras. As coalgebras are the unions of their, finite dimensional subcoalgebras, their representation theory can be viewed as a generalization of the theory of finite dimensional algebras. We will see that many fundamental results extend to coalgebras.

We begin by reviewing some of the most basic definitions and properties of coalgebras and their representations, with the nonspecialist in mind. Some of this material is covered in standard texts [Abe, Mo, Sw], though perhaps in different ways. We mainly follow the treatment in [Gr], with updated terminology. We discuss local finiteness, simple comodules, the coradical filtration and coradically graded coalgebras, and pointed coalgebras in section 2. We proceed in section 3 is to see how the structure theory for finite dimensional algebras extends to coalgebras, with injectives comodules playing a role closely analogous to the role of projectives in module theory. We see that block theory extends to coalgebras, and then discuss the Ext-quivers of coalgebras, and path coalgebras of arbitrary quivers. In a final subsection we describe a special case of the Brauer correspondence for modular coalgebras.

We continue in section 4 by discussing the cohom and cotensor functors, the adjoint pair that dualize hom and tensor for module categories. These functors yield a category equivalence theory for comodules due to Takuechi [Tak], that is now known as *Morita-Takeuchi equivalence*. This in turn allows the construction an equivalent *basic coalgebra* in section 5, which is pointed over an algebraically closed field. The construction of a basic coalgebra first appeared in [Sim], and reappeared [CMo] where it was.studied further. It follows that any arbitrary coalgebra is equivalent to a suitably large subcoalgebra of the path coalgebra of its quiver (5.1). In the hereditary case, we get the entire path coalgebra (5.2) Representations of path colgebras can be regarded as quiver representations that are locally nilpotent (5.3)

Examples drawn from quantum and algebraic group theory are given in section 6 . Quantum and algebraic groups provide an example of a setting where coalgebras and comodules are pertinent. When the base field is infinite comodules correspond to *rational* representations of the (quantum) group. Recent work [CKQ], addresses the transpose and the existence of almost split sequences for comodules. We discuss this work in section 7, and present a special case, which allows for the construction of almost split sequences in the category of finite-dimensional comodules. This result enables the construction of the Auslander-Reiten quiver.

We conclude with a remark from [Tak] characterizing comodule categories among abelian k-categories. The reader may find hypotheses required an abelian k-category to be a comodule category to be suprprisingly mild.

Conventions:

k a fixed base field
$\otimes = \otimes_k$
$C = (C, \Delta, \varepsilon)$ a coalgebra over k
$M = (M, \rho)$ a right C-comodule
${}^B\mathcal{M}^C$ the category of B, C -bicomodules
$\mathcal{M}^C$ the category of right C -comodules
${}^* = Hom_k(\_, k)$
$Hom^C(M, M') = Hom_{\mathcal{M}^C}(M, M')$

2 Coalgebras and Comodules

Definition A *coalgebra* is a vector space C with a comultiplication $\Delta : C \otimes C \rightarrow C$ and a counit map $\epsilon : C \rightarrow k$ satisfying

(a) coassociativity $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$
(b) counitary property $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id_C$.

Thus a coalgebra is obtained by dualizing the associative multiplication map $A \otimes A \rightarrow A$ and unit map $k \rightarrow A$. So a finite dimensional coalgebra is the linear dual of a finite dimensional algebra (and vice-versa). While this duality might lend an air of redundancy to coalgebra theory, there are properties are enjoyed by infinite dimensional coalgebras which are denied infinite dimensional algebras.

Comodules are similarly defined by dualizing the definition of module.

Definition A *right C-comodule* for a coalgebra C is a vector space M with a comodule structure map $\rho = \rho_M : M \to M \otimes C$ satisfying

(a) $(id \otimes \Delta)\rho = (\rho \otimes id)\rho$
(b) $(id \otimes \varepsilon)\rho = id_M$.

We shall assume that comodules are on the right unless we say otherwise.

If M happens to be a subspace of C, then (with $\rho = \Delta$),we have that M is a subcomodule if $\rho(M) \subset M \otimes C$; here M is said to be a *right coideal* of C.

By further analogy, we say that a subspace $D \subset C$ is a *subcoalgebra* if $\Delta(D) \subset D \otimes D$. If $I \subset \ker \varepsilon$ satisfies $\Delta(I) \subset I \otimes C + C \otimes I$, we say that I is a *coideal.*

A linear map between comodules $f : M \to N$ is *a comodule homomorphism* if $(id\otimes f)\rho_M = \rho_N f$. Let $Hom^C(M, N)$ denote the space of comodule morphisms for $M, N \in \mathcal{M}^C$.

The fundamental homomorphism theorems hold as one would guess. The category $\mathcal{M}^C$ of right comodules is an abelian category.

Coalgebras generalize finite dimensional algebras because of the following fact, sometimes known as the

2.1 "Fundamental Theorem of Coalgebras"

Proposition 1 *Every coalgebra is the sum of its finite-dimensional subcoalgebras.*

We shall prove this fact soon, after surveying some notation and results. We follow [Gr] here.

Let X and Y be vector spaces with bases $\{x_j\}$ and $\{y_k\}$ respectively. Let $u \in X \otimes Y$ and express

$$u = \sum u_{ik} \otimes y_k = \sum x_j \otimes v_{ji}.$$

where $u_{ik} \in X$ and $v_{ji} \in Y$ are uniquely determined. Define

$$L(u) = span\{u_{ik}\} \text{ and } R(u) = span\{v_{ji}\}$$

Note that $L(u)$ and $R(u)$ are finite dimensional. The definitions of L and R extend in the obvious way to subsets of $X \otimes Y$ by taking sums. Obviously $L(U)$ and $R(U)$ are finite dimensional if U is.

Now let M be a right C-comodule. It is easy to see that for any subset $U \subset M$, $L(\rho(U))$ is the subcomodule of M generated by U (i.e. the intersection of subcomodules containing U). In particular, if U is a right comodule, then $L(\rho(U)) = U$. These facts follow from the counitary property (b) for comodules.

Now let M be a comodule with basis $\{m_i\}$. Write $\rho(m_i) = \sum_t m_t \otimes c_{ti}$, for some uniquely determined $c_{ti} \in C$ and observe that coassociativity implies that

$$\rho(c_{ij}) = \sum_t c_{it} \otimes c_{tj}.$$

Let $R(\rho(M))$ be denoted by $cf(M)$, which is known as the *coefficient space* of M. It is the subspace spanned by the c_{ti} as just defined and it is evident from the equation just displayed that

1. $cf(M)$ is a subcoalgebra of C .
2. M is a $cf(M)$-comodule that is finite dimensional if M is finite dimensional.

Now for the proof of the Proposition. Let $c \in C$ and let $M = L(\Delta(c))$ be the right subcomodule generated by c. The counitary property $id_C = (\varepsilon \otimes id_C)\Delta$ implies that $c \in cf(M)$. Thus every element of C is contained in a finite dimensional subcoalgebra of C.

We shall denote the subcoalgebra of generated by c by (c).

Remarks: It can be shown that $(c) = R(\Delta(L(\Delta(c)))$, the coefficient space of the right subcomodule generated by c. Also (c) can be expressed as $C^* \rightharpoonup c \leftharpoonup C^*$, using the left and right "hit" actions of C^* (see [Mo]). The coefficient space is a notion dual to the annihilator of module; in fact $cf(M)^\perp$ is the annihilator of M as a left C^*-module.

2.2 Simple Comodules

A comodule is said to be *simple* if it has no proper nontrivial subcomodules. A coalgebra is said to be *simple* if it has no proper nontrivial subcoalgebras. By results above, these simple objects are finite dimensional.

Let S be a simple comodule, and let $D = cf(S)$. Then by dualizing Artinian ring theory, it is not hard to see that:

- D is a simple subcoalgebra of C and
- D is isomorphic as a comodule to the direct sum of $\dim_{End(S)}(S)$ copies of S.

2.3 The coradical filtration

Let $C_0 = corad(C)$ denote the sum of the simple subcoalgebras of C. It is also the socle (=sum of simple subcomodules) of C as a comodule, on either side.

Let C_n be defined inductively by

$$C_n = \Delta^{-1}(C_{n-1} \otimes C + C \otimes C_0)$$

This is known as the *coradical filtration* of C. It is equal to the socle series of C as a comodule. More generally we can describe the socle series of M by $M_0 = \rho^{-1}(M \otimes C_0)$ and

$$M_n = \Delta^{-1}(M_{n-1} \otimes C + M \otimes C_0)$$

Here M_0 equals the direct sum of the simple subcomodules of M (as our terminology suggests).

The next Lemma is a useful property of the degree one term.

Lemma 2 *(see [Mo1], 5.3.1) Let $f : C \to D$ be coalgebra map. Then f is monic if and only if its restriction to C_1 is monic.*

A coalgebra that is an $\mathbb{N}$-graded vector space $C = \oplus C(n)$ is said to be ($\mathbb{N}$-) graded if $\Delta C(n) \subset \sum_i C(i) \otimes C(n-i)$ for all $n \in \mathbb{N}$. Basic results concerning graded coalgebras can be found in [NT]. A graded coalgebra is said to be *coradically graded* [CMu] if $C_0 = C(0)$ and $C_1 = C(0) \oplus C(1)$. The coradical filtration can then be expressed in terms of the grading. Coradically graded coalgebras are a special case of strictly graded coalgebras [Sw], and have recently been generalized in [AS].

Proposition 3 (CMu) *If C is a coradically graded coalgebra, then $C_n = \oplus_{i \leq n} C(i)$ for all $n \in \mathbb{N}$.*

The next result is used in [CMu] to find the coradical filtration for quantized enveloping algebras (see Example 6d below). The statement here is a corrected version of [CMu, 2.3] where the the hypotheses originally only required the coalgebras to be bialgebras.

Proposition 4 ***Proposition 5*** *CMu2] Let C and D be pointed Hopf algebras with the same coradical A. Then $H = C \otimes_A D$ is coradically graded, where $H(m) = \sum_{i=0}^{m} H(i) \otimes H(m-i)$*

2.4 Pointed coalgebras and skew primitives

C is said to be *pointed* if every simple subcoalgebra is of dimension one.

Define the *group-like* elements of C to be

$$G(C) = \{g \in C | \Delta(g) = g \otimes g\}.$$

The kg, $g \in G(C)$ are precisely the one dimensional subcoalgebras, so the span of the $G(C)$ is C_0 if and only if C is pointed. If C is cocommutative ($\Delta = twist \circ \Delta$) and k is algebraically closed, then a coalgebraic version of the Nullstellensatz says that C is pointed. A related fact says that the coordinate Hopf algebra of an affine algebraic group is pointed if and only if it is solvable. Other examples include (quantized) enveloping algebras. See example (d) at the end of this article. $G(C)$ is a group in case C is a Hopf algebra, and thus C_0 is a group algebra.

Assume that C is pointed. We describe how the skew primitive and group-like elements make up the first two terms of the coradical filtration.

We have $C_0 = kG(C)$. Let $g, h \in G(C)$ and set

$$P_{g,h} = \{c \in C | \Delta(c) = g \otimes c + c \otimes h\}$$

The $P_{g,h}$ are called (g, h)-*skew primitives.* Choose a vector space complement $P^{'}_{g,h}$ for $k(g - h)$ in $P_{g,h}$ (the "nontrivial" ones).

The Taft-Wilson theorem (see [Mo1], 5.4.1)states that

$$C_1 = C_0 \oplus \sum_{g,h \in G(C)} P^{'}_{g,h}.$$

The $P_{g,h}$ are called (g, h)-*skew primitives.*

3 Structure theory

3.1 Injectives

Let X be any k-space. We make $X \otimes C$ into a C-comodule via the map $id_X \otimes \Delta$. If M is a comodule, we write $(M) \otimes C$ to denote the comodule with (M) being the underlying vector space of M (whose comodule structure is ignored). This "free" comodule is just the direct sum of $\dim M$ copies of C.

It is known that the category of comodules is a locally finite abelian category (see section 8 below, if interested), and thus has enough injectives. Let's make this more concrete.

Theorem 6 *(a) C is an injective comodule*
(b) $\mathcal{M}^C$ has enough injectives
(c) direct sums and direct summands of injective comodules are injective.

Proof. We prove (a) and (b). Let $f' \in Hom^C(M, C)$ and define $f \in Hom_k(M, k)$ by setting $f = \varepsilon \circ f'$. We can recover f' from f by seeing that $f' = (f \otimes id_C)\rho$. This yields a natural isomorphism between the functors $Hom^C(-, C)$ and $Hom_k(-, k) : \mathcal{M}^C \rightsquigarrow Mod(k)$. Since the latter is exact, so too is the former. This shows that C is injective. Similarly any direct sum of copies of C is injective.

Next we show that every comodule embeds in an injective comodule. To see this consider the map $\rho : M \to (M) \otimes C$. It is straightforward to check that this is an embedding of comodules. For instance the counitary property immediately implies that ρ is a monomorphism. This proves (b).

Remarks: Part (c) of the theorem above might seem surprising since the statement is false for modules in general. Generally, the direct product of comodules does not have a comodule structure.

The category $\mathcal{M}^C$ does not necessarily have enough projectives. If $C = span\{x_i | i = 0, 1, 2, \cdots\}$ is the divided power coalgebra, with

$$\Delta(x_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i}.$$

Then $\mathcal{M}^C$ has no projectives (so it doesn't have enough). Let $C_n = span\{x_i | i = 0, 1, 2, \cdots, n\}$. It is not hard to see that a projective comodule would have $C^* = \lim\limits_{\leftarrow} C_n^*$ as a homomorphic image as a C^*-module. But C^* is a not rational as a C^*-module.

Injective Hulls exist in the "usual" sense: Every comodule M is contained in a maximal essential extension, which is minimal with respect being injective and containing M. This comodule is denoted by $I(M)$. As usual, $I(M) \cong I(M_0)$.The proofs are similar to the module case.

A key Lemma in constructing injective hulls is

Lemma 7 *Let $I \in \mathcal{M}^C$ be injective. If $e_0 = e_0^2 \in End^C(I_0)$, then there exists $e = e^2 \in End^C(I)$ extending $e_{0.}$*

The proof involves inductively constructing idempotents $e_n \in end^C(I_n)$, so that e_{n+1} extends e_n for $n = 0, 1, 2, ...$ This is done by a generalization of Brauer's "famous idempotent lifting procedure". The point here is that the sequence $\{e_n\}$ specifies an idempotent endomorphism of I.

3.2 Indecomposable Injectives

Let $\mathcal{G}$ be a full set of simple comodules in $\mathcal{M}^C$. For each $\mathbf{g} \in \mathcal{G}$, let $m(\mathbf{g})$ denote the multiplicity of $\mathbf{g}$ in C. The coefficient space of $\mathbf{g}$ is a cosemisimple coalgebra and as a right comodule is isomorphic to the direct sum of $m(\mathbf{g}) = \dim_{End(\mathbf{g})} \mathbf{g}$ copies of $\mathbf{g}$. It is known [Gr] that

Theorem 8 *The $I(\mathbf{g})$ form a full set of indecomposable injectives in $\mathcal{M}^C$. As right C-comodules,*

$$C \cong \oplus_{\mathbf{g} \in \mathcal{G}} I(\mathbf{g})^{m(\mathbf{g})}.$$

This generalizes the structure theory for finite-dimensional algebras where injective indecomposables replace porjective indecomposables. We see next that block theory generalizes as well.

3.3 Blocks and Quivers

Define the (*Ext-*) *quiver* of C to be the directed graph $Q(C)$ with vertices $\mathcal{G}$ and $dim_k Ext^1(\mathbf{h}, \mathbf{g})$ arrows from $\mathbf{g}$ to $\mathbf{h}$. Notation as in 3.2 above. The blocks of C are the vertex sets of components of the graph (ignoring directionality) $Q(C)$.

In other words, the blocks are the equivalence classes of the equivalence relation on $\mathcal{G}$ generated by arrows.

Let $bl(\mathbf{g})$ denote the block containing $\mathbf{g}$, so that $\mathcal{G}=\dot{\cup}_{\mathbf{g}\in T} bl(\mathbf{g})$ where T is a set of representatives of blocks of C. In a manner dual to the decomposition theory for finite dimensional algebras, the blocks determine the coalgebra decomposition of C.

Theorem 9 *([Gr]) Let $C(\mathbf{g}) = cf(I(\mathbf{g}))$ Then $C(\mathbf{g}) = C(\mathbf{h})$ for all $\mathbf{h} \in bl(\mathbf{g})$. Also*
(a) $C(\mathbf{g}) \cong \bigoplus_{h\in bl(\mathbf{g})} I(\mathbf{h})^{m(\mathbf{h})}$ as right comodules, and
(b) $C = \oplus_{\mathbf{g}\in T} C(\mathbf{g})$

As a consequence, $C(\mathbf{g})$ is the largest indecomposable subcoalgebra containing $\mathbf{g}$. Also C is indecomposable as a coalgebra if and only if its quiver is (topologically, ignoring directionality) connected.

When C is a Hopf algebra, $G = G(C)$ is a group, $C(k1_G)$ is a Hopf subalgebra and $P := bl(k1_G)$ is a normal subgroup of G. Furthermore, C is isomorphic as an algebra to a crossed product $C(k1_G)\#_t G/P$, see [Mo2].

3.4 Quivers for Pointed Coalgebras

Assume C is pointed. Then the simple subcoalgebras are in bijection with the grouplikes $G = G(C)$. Furthermore, there are $dim_k\ P'_{g,h}$ arrows (see 2.4) from g to h, for all $g, h \in G$. This can be seen by considering the extension

$$0 \to kg \to kg + kd \to kh \to 0$$

corresponding to the (g, h)-skew primitive d. The ordinary $(1, 1)$- primitives for a bialgebra thus correspond to loops.

The enveloping algebra $U(\mathbf{g})$ of a Lie algebra $\mathbf{g}$ is a cocommutative pointed Hopf algebra with quiver consisting of a single vertex and $dim\ \mathbf{g}$ loops.

We shall discuss more examples near the end of this article in section 7.

3.5 Modular Theory

A tool in the computation of the structure of indecomposable injectives for some coalgebras is Green's [Gr1] coalgebraic generalization of the *Brauer correspondence.* Let us summarize this in a nice special case (that is relevant to example 6c below). Let C be a coalgebra defined over a field $\mathbf{K}$, which is the quotient field R, a discrete valuation ring with maximal ideal m. Let $k= R/m$ be the residue field. Assume C contains an R-lattice C_0 with $C = C_0 \otimes_R K$, and write $\overline{C} = C_0 \otimes_R k$.

Now assume that

- C is cosemisimple with $\{V_j |\ j \in \mathcal{J}\}$ a complete set of simple right comodules.

- The simple C-comodules V_j are absolutely simple (i.e., remain simple upon any field extension of $\mathbf{K}$).

Let $\overline{V}_j$ be a specialization of the simple comodule to k (which generally is no longer simple).

Let S_i, $i \in \mathcal{I}$ denote a complete set of simple comodules for C_k and I_i the injective hull of S_i. Let d_{ji} denote the multiplicity of S_i in $\overline{V}_j$ and let c_{ki} denote the multiplicity of S_i in I_k $(i, k \in I)$. The matrices $\mathbf{d} = (d_{ji})$ and $\mathbf{c} = (c_{ki})$ are known respectively as the *decomposition* and *Cartan* matrices. These matrices are infinite in general.

Assume further that

- $End(V_j) = \mathbf{K}$ and $End(S_i) = k$ for all i, j.

J. A. Green's result says (^{t} =transpose):

Theorem 10 $\mathbf{c} = \mathbf{d}^t \cdot \mathbf{d}$

4 Morita-Takeuchi Equivalence

4.1 Cotensor and Cohom

Define the cotensor product $\square_C$ as follows. Let $M \in \mathcal{M}^C$, and $N = (N, \lambda) \in {}^C\mathcal{M}$ and define

$$M \square_C N = \{\alpha \in M \otimes N | (\rho \otimes id_N)(\alpha) = (id_M \otimes \lambda)(\alpha)\},$$

which also can be expressed as the appropriate coequalizer.

Lemma 11 *With $\square = \square_C$ and a subcoalgebra $D \subset C$,*
$M \square C_n \cong M_n$ for all n.
$M \square C \cong M$
$M \square D \cong \rho^{-1}(M \otimes D)$.

The last statement is proved by seeing that the comodule embedding $\rho : M \to M \otimes D$ has image isomorphic $M \square D$. The first two statements are special cases.

Takeuchi solves the problem of determining when there is a left adjoint to $-\square_C N$, $N \in {}^C\mathcal{M}$, see below. The left adjoint functor gives rise to coendomorphism coalgebras that extend (and dualize) the endomorphism rings of finite dimensional modules.

Definitions Let $M, N \in \mathcal{M}^C$ and write $\{N_i\}$ be the directed system of finite dimensional subcomodules of N, ordered by inclusion. Define

$$cohom^C(M, N) = \varinjlim Hom(N_i, N)^*.$$

We write $coend^C(M)$ for $co\hom^C(M, M)$
We say that $M \in \mathcal{M}^C$ is:

- *finitely cogenerated* if M embeds in a finite direct sum of copies of C.
- *quasi-finite* if $Hom(F, M)$ is finite dimensional for finite dimensional $F \in \mathcal{M}^C$.
- *a cogenerator* (for $\mathcal{M}^C$) if C embeds in a direct sum of copies of M.

We note that quasi-finite implies finitely cogenerated.

Proof. It suffices to show that $Hom^C(F, C)$ is finite dimensional for finite dimensional $F \in \mathcal{M}^C$. But we showed in the proof of Theorem 3 that $Hom^C(-, C)$ is naturally isomorphic to $()^* = Hom_k(-, k)$. The assertion follows immediately.

Also, one can observe that M is quasi-finite if and only if $Hom^C(g, M)$ is finite dimensional for simple $g \in \mathcal{M}^C$, which holds if and only if every simple has finite multiplicity in M_0. It is now easy to produce examples of quasi-finite, non-finitely cogenerated comodules.

Adjunction Property: If $X \in {}^B\mathcal{M}^C$ is quasi-finite in M^C, then $-\Box_B X : \mathcal{M}^B \to \mathcal{M}^C$ has left adjoint $cohom^C(X, -) : \mathcal{M}^C \to \mathcal{M}^B$ [Tak, 1.9]. The cohom functor is characterized by the adjunction property.

The following characterizes equivalent comodule categories.

Theorem 12 *([Tak]) Let B, C be coalgebras and let $E \in {}^B\mathcal{M}^C$. The following are equivalent:*
(a) $-\Box_B E : \mathcal{M}^B \to \mathcal{M}^C$ is an equivalence of categories.
(b) E is a quasi-finite injective cogenerator in $\mathcal{M}^C$ and $B \cong coend^C(E)$.

In the case that (a) and (b) are true say that B and C are "(Morita-Takeuchi) equivalent" coalgebras and write $B \sim C$. The inverse functor can be expressed as

$$-\Box_C co\hom(E, C) \approx co\hom(E, -).$$

It is shown [Tak] on the other hand that if $\mathcal{M}^B$ and $\mathcal{M}^C$ are equivalent categories, then the bicomodule E as in the statement of the theorem exists.

Here is a lemma that is useful in reducing things to finite dimensional subobjects.

Lemma 13 *Let D be a subcoalgebra of C and $E \in \mathcal{M}^C$; set $F = \rho_E^{-1}(E \otimes D)$ $(\cong E\Box_C D)$. Then*
(a) E injective implies F injective in $\mathcal{M}^D$.
(b) E finitely cogenerated and D finite dimensional implies F finite dimensional.

A key observation in proving this lemma ([CMo]) is that $Hom^D(Y, E) = Hom^D(Y, F)$ for all $Y \in M^D$.

Remark We can see how $coend(M)$ is the direct limit of coalgebras and so is a coalgebra.. (In [Tak] this is deduced from a universal property of *cohom* instead.) Write E as the directed union of finite dimensional subcomodules E_i as before. Put

$$C_i = cf(E_i)$$
$$F_i = \rho_E^{-1}(E \otimes C_i)$$

as in the lemma above. Then as above,

$$Hom^C(F_i, E) = Hom^C(F_i, F_i);$$

so:

$$cohom(E, E) = \lim_{\to} Hom(E_i, E)^*$$
$$= \lim_{\to} Hom(F_i, E)^*$$
$$= \lim_{\to} End(F_i)^*$$

where the second equality holds because of the cofinality of $\{F_i\}$. Thus $cohom(E, E)$ is a direct limit of finite dimensional coalgebras.

It follows similarly that $E \in {}^{coend^C(E)}\mathcal{M}^C$.

5 Basic Coalgebras

We can write $C \in \mathcal{M}^C$ as a direct sum of indecomposable injectives with multiplicities. Let E denote the direct sum of the indecomposables where each indecomposable occurs with multiplicity one. Clearly E is injective, called a "basic" injective for C. Define

$$B = B_C = coend^C(E).$$

By Theorem 9, B is Morita-Takeuchi equivalent to C. The coalgebra B is *basic* in the sense that

Theorem 14 *([Sim], see also [CMo]) The simple subcoalgebras of B are duals of finite dimensional division algebras.*

As in the remark above, $B = \lim_{\to} End(F_i)^*$. Using the fact that F_i is injective in M^{C_i}, and in fact *basic*, the theorem is proved by reducing to the finite-dimensional case, where the result is known.

Corollary 15 *([CMo]) If* k *is algebraically closed, then B is pointed.*

Remark 16 *In [CG], it is shown that two coalgebras are equivalent if and only if their basic coalgebras are isomorphic. Moreover it is shown that the basic coalgebra of C is $e \rightharpoonup C \leftharpoonup e$ where $e \in C^*$ acts on the left and right by the usual hits. It follows that Morita-Takeuchi equivalence can be expressed with idempotents, just as is the case with artinian algebras. The theory of Morita-Takeuchi equivalence cna be thus simplified.*

5.1 Path Coalgebras

Let Q be a quiver (not necessarily finite) with vertex set Q_0 and arrow set Q_1. The *path coalgebra* kQ of Q is defined to be the span of all paths in Q with coalgebra structure

$$\Delta(p) = \sum_{p=p_2p_1} p_2 \otimes p_1$$
$$\varepsilon(p) = \delta_{|p|,0}$$

where p_2p_1 is the concatenation $a_ta_{t-1}...a_{s+1}a_s...a_1$ of the paths $p_2 = a_ta_{t-1}...a_{s+1}$ and $p_1 = a_s...a_1$ $(a_i \in Q_0)$. Here $|p| = t$ denotes the length of p and the starting vertex of a_{i+1} is the end of a_i.

Thus *vertices* are group-like elements, and if a is an arrow $g \leftarrow h$, with $g, h \in Q_0$, then a is a $(g, h)-$ skew primitive. It is apparent that kQ is pointed with coradical $(kQ)_0 = kQ_0$ and the degree one term of the coradical filtration is $(kQ)_1 = kQ_0 \oplus kQ_1$. More generally, the coradical filtration of kQ is given by

$$(kQ)_n = span\{p||p| \leq n\}.$$

Alternatively, kQ can be defined as the cotensor coalgebra [Ni] associated to the kQ_0, kQ_0-bicomodule kQ_1.

If kQ is finite dimensional, it is the dual of the usual path algebra (or its opposite). Note that if there are infinitely many vertices, the path algebra lacks a identity element; the path coalgebra always has a counit.

Now let C be an arbitrary coalgebra, let B be the associated basic coalgebra and assume k is algebraically closed. Then B is pointed and $Q(C) = Q(B)$. From the universal property of cotensor coalgebras [Ni], it follows that there exists a coalgebra map $B \to kQ(C)$ which is a bijection on the degree one subcoalgebras. By lemma 2, this map is an embedding. Thus we obtain a coalgebraic version (with no finiteness restrictions) of a fundamental result of Gabriel for finite dimensional algebras:

Corollary 17 *([CMo]) Every coalgebra C over an algebraically closed field is Morita-Takeuchi equivalent to a subcoalgebra of $kQ(C)$ containing $kQ(C)_1$*

5.2 Hereditary Coalgebras

A coalgebra C is said to be *hereditary* [NTZ] if homomorphic images of injective comodules are injective. Pointed hereditary coalgebras are exactly path coalgebras kQ for some quiver Q [Ch]. With the aid of the preceding result we also have

Theorem 18 (Ch) *Every coalgebra C over an algebraically closed field is Morita-Takeuchi equivalent to its path coalgebra of $kQ(C)$.*

5.3 Representations of Path algebras

We define a representation of a quiver Q as usual by assigning a vector space V_g to each vertex g and a linear map $f_a : V_g \to V_h$ corresponding to each arrow $a : g \to h$. By composing f's we obtain maps f_p for all nonempty paths p. The category of representations is denoted by $rep(Q)$. It is a standard fact that $rep(Q)$ is equivalent to the category of modules over the path algebra of Q (at least when Q is finite).

Let C denote the path coalgebra kQ, and let M be a C-comodule. To M we associate an object of $rep(Q(C)) = rep(Q)$ as follows. Let I be the coideal spanned by paths of length at least 1, and define $\pi : C \to C/I \cong C_0$ as the projection. Set $\rho_0 = (id_M \otimes \pi) \circ \rho$. Define

$$V_g = \{v \in M | \rho_0(m) = m \otimes g\}$$

and define $f_p : V_g \to V_h$ for paths p (from g to h) by

$$\rho(v) = \sum f_p(m) \otimes p.$$

where the sum is over paths p starting at g. Since the paths are linearly independent, the f_p are well-defined and we get a representation of Q. Note that the sum has only finitely many nonzero terms. Checking details, we find that $\mathcal{M}^{kQ}$ corresponds to the subcategory of "locally nilpotent" representations.

Proposition 19 *(see [CKQ]) For any quiver Q, $\mathcal{M}^{kQ}$ is equivalent to the full subcategory of $rep(Q)$ consisting of representations (V, f) such that for all $g \in Q_0$ and $v \in V_g$, $f_p(v) = 0$ for all but finitely many paths p (starting at g).*

Let Q be the quiver with a single vertex g and a single loop a. Let $V_g = k$ and $f_a = \lambda \cdot$, $\lambda \in k$. This representation does not correspond to a kQ-comodule unless $\lambda = 0$ (though of course it is a module over the path algebra $k[a]$).

We would like two mention the articles [Sim2, NS] that contain interesting results concerning certain types of path coalgebras.

6 Examples

a. Let C be the coordinate Hopf algebra of an affine connected simply connected semisimple algebraic group G or the quantum variant where q is not a root of 1 (see [CP]). If k is of characteristic zero, then C is cosemisimple. So the blocks are singletons.

b. Let C be the coordinate algebra of $SL(2)$. If k is of positive characteristic zero, then C is cosemisimple. The blocks are infinite and infinite in number. The quiver structure is somewhat complicated, and are given by "p-reflections" see [Cl].

c. Let $C = k_\zeta[SL(2)]$ be the q-analog of the coordinate algebra of $SL(2)$ (see [CP]), where q is specialized to a root of unity of odd order l. Assume k is of characteristic zero. The quiver structure is given as follows: For each nonnegative integer r, there is a unique simple module $L(r)$ of highest weight r (closely analogous to the simple highest weight modules for nonquantum groups). These comodules exhaust the simple comodules.

Write

$$r = r_1 l + r_0$$

where $0 \leq r_0 < l$. Define an "l-reflection" $\tau : \mathbf{Z} \to \mathbf{Z}$ by

$$\tau(r) = (r_1 - 1)l + l - r_0 - 2$$

if $r_0 \neq l-1$, and $\tau(r) = r$ if $r_0 = l-1$. Put $\sigma = \tau^{-1}$ (perhaps after checking that τ is a bjiection).

Theorem 20 *([CK]) Set $I(r) = I(L(r))$ for all integers $r \geq 0$.*
(a) If $r_0 = l-1$, then $I(r) = L(r)$.
(b) If $r < l-1$, then $I(r)$ has socle series with factors

$$L(r), L(\sigma(r)), L(r).$$

(c) If $r > l$ (and $r_0 \neq l-1$), then $I(r)$ has socle series with factors

$$L(r), L(\sigma(r)) \oplus L(\tau(r)), L(r).$$

This determines the quiver as having vertices labelled by nonnegative integers and with an arrow $r \leftrightarrows s$ in case $r_0 \neq l-1$ if and only if $s = \tau(r)$. In case $r_0 = l-1$, the block of r is a singleton (equivalently $L(r)$ is injective). Thus the nontrivial block containing $L(r)$, $r < l-1$ has quiver

$$r \leftrightarrows \tau(r) \leftrightarrows \tau^2(r) \leftrightarrows \cdots$$

The injectives indecomposable comodules are all finite dimensional, in contrast to the injectives for the ordinary (nonquantum) modular coordinate coalgebra. Also, the result shows that the coradical filtration is of length 2 (i.e. $C = C_2$). It follows that C is *semiperfect* (see [Lin]) in the sense that every finite dimensional comodule has a projective cover.

These results are obtained using the modular result at the end of section 3. We sketch here a short proof of (b) and (c) above. The simple comodules for $C = k_\zeta[SL(2)]$ are known to be highest weight modules, say with weight $r \in \mathbb{N}$, which can be specialized to C. The resulting $C-$comodules are the simple ones $L(r)$ in case $r_1 = l-1$ or $r < l$; otherwise they have composition series $L(\rho(r)) \leq L(r)$. [CK]. Fixing $r < l-1$, we can list the simples of the nontrivial block containing $L(r)$ as the comodules $S_i = L(\tau^i(r))$, $i \in \mathbb{N}$. Thus the decomposition matrix $\mathbf{d} = (d_{ij})$ for this block is given by $d_{ii} = 1 = d_{i+1,i}$, and 0's elsewhere. By the Brauer correspondence in section 3, the Cartan matrix $\mathbf{c} = (c_{ij})$ is given by $c_{ii} = 2$ and $c_{i+1,i} = c_{i,i+1} = 1$, and 0's elsewhere. Now the conclusions (b) and (c) follow once we know the there are no self-extensions of simples. This fact follows by following the argument or the nonquantum modular theory, e.g. [Ja, 2.14].

In [CK], we also determine the indecomposable injectives for the standard quantum analogs of coordinate coalgebras of 2×2 matrices and general linear group at odd roots of unity. The situation is more complicated for these coalgebras.

d. Let C be the quantized enveloping algebra $U_q(\mathbf{g})$, associated to a finite dimensional complex simple Lie algebra $\mathbf{g}$ of rank n, defined over a field k of characteristic zero. See [Lu1] and e.g., [Mo], [CP] for definitions.

Assume q is specialized to an element of k which is not a root of unity. C is generated as an algebra by group-likes $K_i^{\pm 1}$ and the $(K_i, 1)-$ skew primitives E_i and K_iF_i, $i = 1, ..., n$. C is pointed with $G(C)$ being a free abelian group of rank n. It turns out that the only skew primitives are the "obvious" ones, spanned by gE_i , gF_i , and the "trivial" ones $g-h$, $(g, h \in G(C))$. (The nonsimple root vectors are not skew-primitive.) Thus the vertex set is $G(C)$, and arrows are

$$gK_i \rightrightarrows g,$$

for any $g \in G(C)$, $1 \leq i \leq n$.

This result was obtained in [CMu] when q is transcendental over the rationals. Further progress was made (working with quantized coordinate algebras) in [Mus]. E. Müller [Mü] solved the problem of determining the coradical filtration more generally, including versions for specializations to roots of unity. His methods rely on Lusztig's newer construction of quantized enveloping algebras [Lu]. Another recent broad generalization appears in [AS].

7 Almost Split Sequences

In [CKQ], we investigated the existence of almost split sequences for comodules given a fixed right or left-hand term. To construct the Auslander-Reiten quiver, one needs to be able to iterate the construction in some subcategory.

A coalgebra is defined to be *right semiperfect* [Lin] if every simple *left* C-comodule has an finite-dimensional injective hull (equivalently, every finite-dimensional right comodule has a projective cover). In the context of group schemes, (left and right) semiperfect coalgebras were called *virtually linearly reductive* in [D1,D2].

Theorem 21 (CKQ) *Let C be a right semiperfect coalgebra such that soc(I(S)/S) is of finite length for all simple right C-comodules. Then the category of finite-dimensional right C-comodules has almost split sequences.*

A coalgebra is defined to be *right semiperfect* [Lin] if every simple *left* C-comodule has an finite-dimensional injective hull (equivalently, every finite-dimensional right comodule has a projective cover). In the context of group schemes, (left and right) semiperfect coalgebras were called *virtually linearly reductive* in [D1,D2]. The theorem applies to right semiperfect coalgebras whose Ext-quivers have only finitely many arrows ending at each vertex. Special cases of this are coalgebras whose Ext-quivers have only finitely many arrows and finitely many paths ending at each vertex. Hence, finite-dimensional comodules over subcoalgebras of path coalgebras of such quivers have almost split sequences.

The theorem applies to $k_\zeta[SL(2)]$, which is semiperfect, as in 6c above, (though not to the path coalgebra of its quiver). More generally, let $k_\zeta[G]$ be a quantized coordinate algebra at the root of unity ζ as in [APW]. Then by [loc. cit., section 9] (see also [AD] for a different proof), $k_\zeta[G]$ is a semiperfect coalgebra. It should be noted that a Hopf algebra is semiperfect as a coalgebra if and only if it is right semiperfect.

8 Abelian Categories

We append a remark from [Tak].

Let $\mathcal{A}$ be an abelian k-category that

- has exact directed colimits and has a set of objects of finite length which generate $\mathcal{A}$ (i.e. $\mathcal{A}$ is "locally finite"), and such that
- $\mathcal{A}(S,S)$ is finite dimensional for all simple objects S.

Takeuchi [Tak] says such categories $\mathcal{A}$ are of "of finite type". He shows that

Theorem 22 *An abelian* k*-category $\mathcal{A}$ is equivalent to a comodule category $\mathcal{M}^C$ for some coalgebra C if and only if $\mathcal{A}$ is of finite type.*

The proof mirrors the construction of the basic coalgebra. Let E be a an injective cogenerator, which is the direct sum of injective objects, each occurring with multiplicity one. Then E is isomorphic to the direct limit of its finite length subobjects E_i. Letting $C = \varinjlim \mathcal{A}(E_i, E)^*$ works.

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