A Guide to Writing Mathematics

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Introduction

This is a math class! Why are we writing?

There is a good chance that you have never written a paper in a math class before. So you might be wondering why writing is required in your math class now.

The Greek word *mathemas*, from which we derive the word mathematics, embodies the notions of knowledge, cognition, understanding, and perception. In the end, mathematics is about *ideas*. In math classes at the university level, the ideas and concepts encountered are more complex and sophisticated. The mathematics learned in college will include concepts which cannot be expressed using just equations and formulas. Putting *mathemas* on paper will require writing sentences and paragraphs in addition to the equations and formulas.

Mathematicians actually spend a great deal of time writing. If a mathematician wants to contribute to the greater body of mathematical knowledge, she must be able communicate her ideas in a way which is comprehensible to others. Thus, being able to write clearly is as important a mathematical skill as being able to solve equations. Mastering the ability to write clear mathematical explanations is important for non-mathematicians as well. As you continue taking math courses in college, you will come to know more mathematics than most other people. When you use your mathematical knowledge in the future, you may be required to explain your thinking process to another person (like your boss, a co-worker, or an elected official), and it will be quite likely that this other person will know less math than you do. Learning how to communicate mathematical ideas clearly can help you advance in your career.

You will find that writing good mathematical explanations will improve your knowledge and understanding of the mathematical ideas you encounter. Putting an idea on paper requires careful thought and attention. Hence, mathematics which is written clearly and carefully is more likely to be correct. The process of writing will help you learn and retain the concepts which you will be exploring in your math class.

What does good mathematical writing look like?

As you learn more math, being able to express mathematical ideas will become more important. It will no longer be sufficient just to be able to write down some final "answer". There is a good reason why Herman Melville wrote *Moby Dick* as a novel and not as the single sentence:

> *The whale wins.*

For this same reason, just writing down your final conclusions in an assignment will not be sufficient for a university math class.

You should not confuse writing mathematics with "showing your work". You will not be writing math papers to demonstrate that you have done the homework. Rather, you will be writing to demonstrate how well you understand mathematical ideas and concepts. A list of calculations without any context or explanation demonstrates that you've spent some time doing computations; however, a list of calculations without any explanations omits ideas. The ideas are the mathematics. So a page of computations without any writing or explanation *contains no math*.

When you write a paper in a math class, your goal will be to communicate mathematical reasoning and ideas clearly to another person. The writing done in a math class is very similar to the writing done for other classes. Your are probably already used to writing papers in other subjects like psychology, history, and literature. You can follow many of the same guidelines in a mathematics paper as you would in a paper written about these other subjects.

Basics: Combining Words and Equations

Following the rules of grammar.

Good writing observes the rules of grammar. This applies to writing in mathematics papers as well! When you write in a math class, you are expected to use correct grammar and spelling. Your writing should be clear and professional. Do not use any irregular abbreviations or shorthand forms which do not conform to standard writing conventions. Mathematics is written with sentences in paragraphs. (And yes, paragraphs are important. It is not amusing to read a three-page paper consisting of just one paragraph.)

There is however one element in mathematical writing which is not found in other types of writing: formulas. However, it may surprise you to know that in a math paper, formulas and equations follow the standard grammatical rules that apply to words. Mathematical

symbols can correspond to different parts of speech. For instance, below is a perfectly good complete sentence.

$$1 + 1 = 2.$$

The symbol "=" acts like a verb. Below are a couple more examples of complete sentences.

$$3xy < -2.$$
$$5z \in \mathbb{R}.$$
$$9 - s \neq t.$$

Can you identify the verbs? On the other hand, an expression like

$$5x^2z - 10y$$

is not a complete sentence. There is no verb. Such an expression should be treated as a noun. Can you identify the nouns in the previous examples?

Formulas and equations need to be contained in complete sentences with proper punctuation. Here is an example:

> The total revenue, R, made from selling widgets is given by the equation
>
> $$R = pq,$$
>
> where p is the price at which each widget is sold and q is the number of widgets sold. Based on past experience, we know that when widgets are priced at \$15 each, 2000 widgets will be sold. We also know that for every dollar increase in price, 150 fewer widgets are sold. Hence, if the price is increased by x dollars, then the revenue is
>
> $$\begin{aligned} R &= (15 + x)(2000 - 150x) \\ &= -150x^2 - 250x + 30,000. \end{aligned}$$

Notice how punctuation follows each of equations. A computation which ends a sentence needs to end with a period. Computations which do not end sentences are followed by commas.

A good way to improve your mathematical writing is by reading your writing, *including all of the equations*, out loud. Your ears can often pick out sentence fragments and grammatical errors better than your eyes. If you find yourself saying a series of fragmented sentences and equations, you should do some rewriting.

There are a couple of other important things to observe in the above example. Notice how "we" is used. The use of first person is common in mathematics, especially the plural "we", so don't be afraid to use the word "we" in the papers you write in your math class.

Another thing to notice is that important or long formulas are written on separate lines. You can make your mathematical writing easier to read if you place each important formula on a line of its own. It's hard to pick out the important formulas below:

> If d is Bob's distance above the ground in feet, then $d = 100 - 16t^2$, where t is the number of seconds after Bob's Flugelputz-Levitator is activated. Solving for t in the equation $100 - 16t^2 = 0$, we find that $t = 2.5$. Bob hits the ground after 2.5 seconds.

This is clearer:

> If d is Bob's distance above the ground in feet, then
>
> $$d = 100 - 16t^2,$$
>
> where t is the number of seconds after Bob's Flugelputz-Levitator is activated. Solving for t in the equation
>
> $$100 - 16t^2 = 0,$$
>
> we find that $t = 2.5$. Bob hits the ground after 2.5 seconds.

Symbols and words.

It is important to use words and symbols appropriately. Part of being able to write mathematics well is knowing when to use symbols and knowing when to use words.

Don't use mathematical symbols when you really mean something else. A common mistake is to misuse the "=" symbol. For instance:

$$3^{2x} - 2(3^x) = -1 \;=\; (3^x)^2 - 2(3^x) + 1 = 0 \;=\; (3^x - 1)^2 = 0 \;=\; 3^x = 1 \;=\; x = 0.$$

!!

Do not use the equal sign when you really mean "the next step is" or "implies". The above example is really saying that $-1 = 0 = 1$! Using arrows instead of equal signs is a slight improvement, but still not desirable:

$$3^{2x} - 2(3^x) = -1 \;\rightarrow\; (3^x)^2 - 2(3^x) + 1 = 0 \;\rightarrow$$
$$(3^x - 1)^2 = 0 \;\rightarrow\; 3^x = 1 \;\rightarrow\; x = 0.$$

With a sequence of calculations, sometimes it is best to just place each equation on a separate line.

$$\begin{aligned} 3^{2x} - 2(3^x) &= -1 \\ (3^x)^2 - 2(3^x) + 1 &= 0 \\ (3^x - 1)^2 &= 0 \\ 3^x &= 1 \\ x &= 0. \end{aligned}$$

For a difficult computation where the reader might not readily follow each step, you can include words to describe the steps you take.

> We want to solve for x in the equation
>
> $$3^{2x} - 2(3^x) = -1.$$
>
> We can rewrite this equation in terms of 3^x:
>
> $$(3^x)^2 - 2(3^x) + 1 = 0.$$
>
> After factoring, this becomes
>
> $$(3^x - 1)^2 = 1$$
>
> and it follows that $3^x = 1$, or $x = 0$.

However, make sure that your paper has a single flow. Don't explain a calculation using the "two-column method".

$3^{2x} - 2(3^x) = -1$	Solve this equation.
$(3^x)^2 - 2(3^x) + 1 = 0$	Collect the terms on one side.
$(3^x - 1)^2 = 0$	Factor.
$3^x = 1$	Use the Zero Factor Property.
$x = 0$	Solve for x.

This is hard to read through. It's also bad style.

Some things are best expressed with words. But other things are best expressed with mathematical notation. For instance, it hard to read:

> It follows that x plus two is larger than zero.

Here, mathematical notation is more appropriate.

> It follows that $x + 2 > 0$.

Miscellaneous comments.

Here are a couple of other pointers to help you get started with your mathematical writing.

- Don't start a sentence with a formula. While it may be grammatically correct, it looks strange.

  > $t = 5$ when $w = 2000$, so we can conclude that the new factory will be completely overrun with cockroaches in 5 years.

  > f is globberfluxible at $x = 3$.

  Adding just a word or two can fix these examples.

  > Since $t = 5$ when $w = 2000$, we can conclude that the new factory will be completely overrun with cockroaches in 5 years.

  > The function f is globberfluxible at $x = 3$.

- Don't turn in pages of unreadable scribbles to your professor. In college, papers are typed. They are also usually double-spaced with large margins. Mathematics papers adhere to the same standards as papers written for other classes.

- While it is a good idea to type your paper, you may have to leave out the formulas and insert them by hand later. It is perfectly acceptable to write formulas by hand in a math paper. Just make sure that your mathematical notation is legible. If you do decide to type the equations, please be aware that variables in equations and formulas are usually italicized (to set them apart from the text). Many word processing programs contain equation editors. In newer versions of Microsoft Word, the equation editor is available under the `Insert` menu. Select `Object...`, and then `Equation`.[1] If you are going to be writing a lot of technical documents, it might be worthwhile to learn TEX or LATEX. These are professional mathematical typesetting languages. This document was written with LATEX. You may also find satisfactory results typing papers in Maple or some other mathematically oriented software program.

- Use mathematical notation correctly. As you learn to write more complicated formulas, it is all too easy to leave out symbols from formulas. Learn how to use symbols properly!

- Use language precisely and correctly. Make sure that the words you use really mean what you think they mean. Mathematics requires very precise use of language. Another thing to avoid is overuse of the word "it". Mathematical papers with a lot of pronouns like "it" and "that" tend to be hard to read. It is often hard for the reader to see what "it" is referring to. If you, the author, are also having difficulty seeing what "it" is referring to, then you may be having some difficulty with the mathematical ideas; you may need to think more about the ideas you are writing about.

- Try to write as simply and directly as possible. No one likes to read ponderous pretentious prose.

[1] In Microsoft Word, it is also possible to place a button on the tool bar which activates the equation editor. Select `Configure...` beneath the `Tools` menu. In the window that pops up, select the `Commands` tab. Under the `Insert` category you will find the `Equation Editor` command. Drag the equation editor icon to the tool bar.

Mathematical Ideas into Writing

Organizing your paper.

A well-organized paper is easier to read than a disorganized one. Fortunately, there are some standard ways to order a mathematics essay.

First, there is some type of introduction. Usually, the introduction states the problem. Even if you are answering a problem from a text book, you should not assume that the reader is familiar with the text book or even has a copy of the text book available to him or her. However, do not just copy the problem! You must rewrite the problem in your own words.

A good introduction should also discuss the significance of the problem. The introduction is where you will need to "hook" the reader.

It is not a bad idea to also preview the rest of the paper in the introduction. Give the reader some idea of what to expect later.

> We will analyze the revenue using a linear model and then examining the graphs generated by the model.

> The production of fava beans will be modeled using a C program.

> First, we will analyze the population using numerical methods. Then, we will analyze the population using formulas. We will then compare the two different results.

Some papers then state the "answer" to the problem right after the introduction. Other papers place the "answer" at the end. This is a matter of taste. Sometimes, the end result is the most important thing in the paper. You may need to place the end result at the beginning to entice the reader. On the other hand, sometimes the method of arriving at the end result is more important. In such a case, putting the result at the end may be more sensible.

In any case, it is best to state the result in terms of the original problem using real-world terms.

> The solution is $t = 6$.

The solution to the equation is $t = 6$. The population of Utopia is at its smallest 6 years after the plague begins.

Make sure that the arguments you write are carefully organized. It may help you to write an outline before you begin writing a mathematics paper. Writing an outline will also help you think about the concepts more clearly and thus will help you learn the material. As you write about more advanced mathematical problems, organization will become even more important.

Writing for your audience.

For most papers that you write in your math class, you should assume that the reader has about the same mathematical knowledge that you have. When you write up the solution to a homework problem, it might be helpful to think that you are writing to a student in another section of the same class or in a similar class at another school. Some of the papers you will be writing will be directed toward a reader who may know *less* math. The purpose of a math paper is not just to show the professor that you know something. Your math professor already knows the subject; you are not writing for his or her benefit. You are writing for someone who doesn't know the subject. (That someone may be you! You can use your writing assignments to help review for exams.)

In your mathematics writing, you will be communicating to the reader *why* and *how* you arrived at a solution. You will also want to convince your reader that your particular reasons and your particular means to the solution are correct. A good mathematical paper not only should provide clear explanations, but should also be able to persuade a skeptical reader.

Many times, if you can arrive at the same solution through alternate routes, you can make your writing more persuasive. You may want to analyze a problem using both computers and algebra. Or you might compare a graph with real-world information. Pictures and graphical depictions can be very helpful for your reader.

Specific examples will also help to make your writing more persuasive. You can help a reader understand an abstract general argument by showing how the argument applies to a specific case. You can also use "extreme" cases to show the limits of an argument.

Make sure that what you write is relevant to the problem. Including extraneous comments or information demonstrates a lack of understanding of the ideas and concepts, and reduces the overall effectiveness of your mathematical writing. Thinking about the reader will help you to decide which details you need to include and which details you should leave out. Calculations which are tedious and uninteresting to the reader can be readily omitted. (Again, mathematics writing is not the same as showing work. You don't

need to show everything.) The reader of a college mathematics paper will probably not be interested in reading how to multiply 5 and 74. Leave out what is unimportant. On the other hand, don't leave out anything which is critical to the key ideas you are trying to explain. Learning what is important and what is unimportant will help you understand mathematics better.

You should not assume that the reader is familiar with the problem you are solving. While you do not need to restate the problem in its entirety, be sure to give an overview of all important details in the problem. You also should not assume that the reader is in the same mind set as you. In your writing, state any assumptions which you have made. For instance, in physics problems, it is often assumed that everything is frictionless. But just because this assumption is made nearly all the time doesn't mean that your reader will automatically make this assumption; your reader may not be familiar with physics. Just because you assume something is true doesn't mean that your reader will. So write it down!

Defining variables and formulas.

Quantities and functions can be, and often should be, represented with letters. However, the letters which are chosen are arbitrary. You should explicitly state what all letters in your formulas represent in as precise a manner as possible. For instance:

Either n or $n+1$ is even. 

What is n? If $n = 8.5$ is the above statement true? A better way of stating this is:

For any whole number n, either n or $n+1$ is even. 

A common phrase used in mathematics is "Let...".

Let x be any real number. 

Let P be the population of Los Angeles in 2010. 

Let $f(x) = x^2 + 1$. 

In the last example, x is a place holder. It doesn't require a proper introduction. However, it would be better to write:

> Let $f(x) = x^2 + 1$ for all real numbers x. 

If describing all the variables gets tedious, try not assigning any variables at all. The following example clearly needs improvement.

> The volume is ℓwh. 

The following example is adequate, but wordy.

> The volume of the box is ℓwh, where ℓ is the length, w is the width, and h is the height.

We can write this most elegantly by removing the variables.

> The volume of the box is the product of the length, the width, and the height.  !

You need to be especially careful with variables representing real-world quantities. Avoid describing them vaguely, as in:

> Let $D(t)$ be the distance at a time t. 

Including units would make this clearer, but the description is still vague.

> Let $D(t)$ be the distance in miles at t hours.

Try to be as specific as possible.

> Let $D(t)$ be Agnes's distance from the arena in miles t hours after the riot began. 

Also, be careful that each symbol you use represents only one thing. This can actually be more subtle than it sounds. The following example seems to be rather clear.

Let P be the escaped wombat population (in thousands) t years after 1990 and suppose that

$$P = 0.5(1.12)^t.$$

The wombat population in 1992 is approximately 672. We can see this by setting $t = 2$ and observing that

$$P = 0.5(1.12)^2 = 0.6272 \text{ thousand wombats.}$$

If we want to predict when the wombat population will reach 2000, we set $P = 2$ and solve for t using logarithms.

$$\begin{aligned} 2 &= 0.5(1.12)^t \\ \log 2 &= \log 0.5 + t \log 1.12 \\ t &= \frac{\log 2 - \log 0.5}{\log 1.12} \approx 12.23 \text{ years.} \end{aligned}$$

The wombat population will reach 2000 in the year 2002.

I think that the above example would be considered unobjectionable by most readers. It looks very clear and understandable. The variable P is always standing for the wombat population. However, notice that in the first paragraph, P is the wombat population in general. In the next paragraph, $P = 0.6272$, the wombat population in 1992. And in the last paragraph, $P = 2$. The meaning of P appears to be changing every time that it is used. In the first paragraph, P represents the population at *any* time. In the other instances, P represents the population at *one particular* time. The problem can be fixed omitting some variables and adding others.

Let P be the escaped wombat population (in thousands) t years after 1990 and suppose that

$$P = 0.5(1.12)^t.$$

By substituting 2 for t in the above equation, we can see that in 1992, the wombat population is approximately 672.

$$0.5(1.12)^2 = 0.6272 \text{ thousand wombats.}$$

Let t_{2000} be the year when the wombat population reaches 2000. Then,

$$\begin{aligned} 2 &= 0.5(1.12)^{t_{2000}} \\ \log 2 &= \log 0.5 + t_{2000} \log 1.12 \\ t_{2000} &= \frac{\log 2 - \log 0.5}{\log 1.12} \approx 12.23 \text{ years.} \end{aligned}$$

The wombat population will reach 2000 in the year 2002.

While in the above example, we can afford a little bit of sloppiness with the variables, in more complex problems, this could be a source of potential trouble. When a symbol is used to represent two different things (even, or perhaps especially, if those things are similar), the reader (and the writer!) can become confused. A symbol used in two different ways is not only confusing, but often results in incorrect mathematics!

Just as variables need to be introduced carefully, also be sure not to pull formulas out of thin air. Tell the reader how you get each formula or what each formula means. It's not very pleasant to get hit with formulas without any warning.

Using pictures in mathematics.

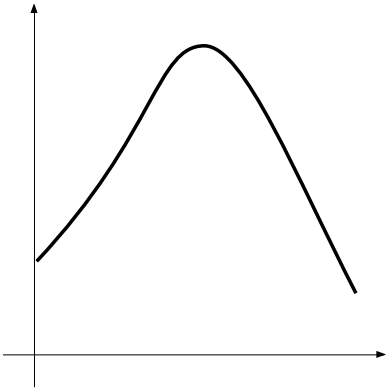
A picture can really be worth a thousand words. I strongly encourage you to use visual arguments in your mathematical writing. However, if you do include a picture, a diagram, a graph, or some other visual mathematical representation, make sure that you fully explain how it fits into your mathematical argument.

Looking at the graph, we can see that the result is true.

What should the reader look for in the graph? Why does the graph support the argument? Be more specific.

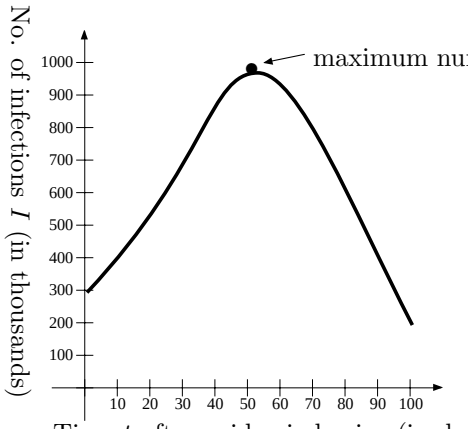
The graph increases sharply at $t = 3$, confirming our earlier prediction that the robots will begin a homicidal rampage three years from now.

A good graph should convey relevant and specific information to the reader. The following graph is vague.



Graphs and diagrams need to be neatly drawn and clearly labeled. Indicate the scale on the axes. You should point out significant graphical features.

Cooties infections versus time



If you draw a graph by hand, use a straight edge. You may want to generate your graphs using a computer. Be careful though. Programs like Excel or Microsoft Office generally are not good at generating mathematical graphs. You will more likely have success using a math program like Maple.

Any diagrams you draw should also be carefully labeled. Be sure to label everything that you refer to in your argument.

Epilogue

Writing mathematics is not the easiest thing to do. Writing mathematics is a skill which takes practice and experience to learn. There are many resources here at Purdue Calumet which are available to you to help you with your mathematical writing. Among these are the Math Lab and the Writing Lab.

If you have not written mathematics much before, it may feel frustrating at first. But learning to write mathematics can only be done by actually doing it. It may be hard at first, but it will get easier with time and you will get better at it. Do not get discouraged! Being able to write mathematics well is a good skill to learn, and one which you will keep for a lifetime.

A mathematical writing checklist

Below is a checklist which will help you follow the guidelines outlined above in your mathematical writing.

- ☐ **Is your paper neatly typed?**

  If you write the equations by hand, make sure that you have written in *all* of the equations. Also make sure that you have included all of the diagrams and graphs you intended to. Make sure that the paper is double-spaced and has wide enough margins.

- ☐ **Has the paper been proofread?**

  In college, sloppy work is not appreciated. Do check over everything.

- ☐ **Is there an introduction?**

  Make sure that you explain the problem to the reader. Assume that the reader is unfamiliar with the problem. The introduction should also try to indicate to the reader why the problem is interesting and give some indication of what will follow in the paper.

- ☐ **Did you state all of your assumptions?**

  Write down any physical assumptions that you made. (Did you assume that there was no friction? That the population grew with unlimited resources? That interest rates remained steady?) Write down any mathematical assumptions that you made. (Did you assume that the function was continuous? Linear? That x was a real number?)

- ☐ **Are the grammar, spelling, and punctuation correct? Is the writing clear and easy to understand?**

  Make sure that there are no sentence fragments. The formulas and equations too need to be contained in complete sentences. Equations and formulas (and the words too) should have correct punctuation as well. Make sure that your paper flows smoothly and reads well. And please, don't be careless! Check your spelling!

- ☐ **Are all of the variables defined and described adequately?**

  Make sure that you introduce each variable that you use. Describe each variable as precisely as possible. Don't forget any units!

- ☐ **Are the mathematical symbols used correctly?**

  Don't use an "=" sign outside of a formula. Make sure that the symbols are not misused. Use equations and formulas where they are appropriate.

☐ **Are the words used correctly and precisely?**

Avoid using vague language and too many pronouns. Use words where they are appropriate.

☐ **Are the diagrams, tables, graphs, and any other pictures you include clearly labeled?**

Graphs should be drawn with a straight edge (or computer-generated) with axes clearly labeled (with units if appropriate) and the scale indicated. Diagrams should be neatly drawn with relevant labels.

☐ **Is the mathematics correct?**

This should be obvious.

☐ **Did you solve the problem?**

Sometimes in all of the fuss, people forget to answer the problem. Do answer the question! Also, see if you can write the solution in "real-world" terms.