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The Use of Logic in Teaching Proof

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Introduction: Even rather simple proofs and disproofs are built atop a normally unexpressed substructure of great logical and linguistic complexity. For example, in **[7]** I described a number of the many reasoning processes needed to establish the truth or falsity of the following statements: (1) The square of any rational number is rational; (2) For all real numbers a and b, if $a > b$ then $a^2 > b^2$; and (3) For all real numbers x, if x is irrational then $-x$ is irrational. The article cites evidence that a significant number of students taking college mathematics courses do not bring with them an intuitive feeling for the logic required to evaluate such statements and argues that some explicit instruction in logical reasoning is needed in courses that require students to engage in proof writing. However, because proofs and disproofs of even elementary statements require a substantial base of understanding, a "clarifying" analysis for a proof may be so complex that if students could understand it, they would not need it in the first place. Figuring out how to present proof construction simply enough to be intelligible yet detailed enough to be effective is one of an instructor's greatest challenges.

The following sections contain ideas for helping students learn to construct simple proofs and disproofs. Most are approaches I have used myself and for which student reaction has been positive. Others are ideas for which colleagues have reported success.

For students who come to a course with reasonably good intuition for logical principles, merely seeing them stated and working a few examples can be a pleasure – like the delight of the Molière character who learned one day that he'd been speaking prose all his life. For many students, however, simple exposure to principles of logic is not sufficient to counteract deeply ingrained incorrect patterns of thought, and follow-up instruction is needed to illustrate the uses of the principles in mathematical contexts. Thus Section I contains not only suggestions for how to take advantage of having provided students with a brief introduction to basic principles of logical reasoning before requiring them to make serious attempts at mathematical proof but also advice for how to help students develop a firmer and deeper grasp of reasoning principles as proof and disproof of various mathematical topics are discussed. Section II offers additional strategies to guide students through their initial proof efforts and lead them to see the desirability of expressing proofs with care, and Section III discusses additional ways to help students come to learn the need for proof.

I. Building on Initial Coverage of Logical Principles

Using Puzzle Problems: To make the transition from elementary logic to proof, it can be helpful to assign puzzle problems, such as Raymond Smullyan's knights and knaves.**[20]** These puzzles posit an island where each inhabitant is either a knight, who always tells the truth, or a knave, who always lies, but it is impossible to distinguish knights from knaves by their appearance. Each puzzle describes a situation in which certain inhabitants make certain statements, and the goal is to figure out who is lying and who is telling the truth. When solutions are discussed in class, quite a number of students make it clear that they do not have a natural feeling for the kind of indirect reasoning needed to solve most of the puzzles. Nonetheless, almost all students seem to enjoy the puzzles, and working on them helps develop a basis of intuition for proof by contradiction. Discussing the solutions serves to illustrate how inference rules are used in practice and helps students develop a sense for the flow of deductive reasoning, which they will use later in mathematical proofs of all types.

Using Natural Deduction Proofs: John Barwise and John Etchemendy developed computer software called Tarski's World, named after the logician Alfred Tarski, to represent situations in a world that consists of a grid containing a number of geometric shapes in a variety of positions. The accompanying courseware **[2,3]** shows students how, among other things, to produce natural deduction proofs of statements about the shapes in the world. Work of Lee and Stenning **[11]** supports the view that use of these materials improves students' ability to reason deductively. Another teacher who uses instruction in natural deduction to prepare students for reasoning in more general environments is Richard L. Morrow, a middle school mathematics coordinator with advanced training in logic.**[16]** When Morrow first taught geometry to a group of gifted eighth graders, his students finished the book a month before the end of the year. Thinking to fill in the extra time, Morrow began the course the next year with a few weeks study of formal logic, focusing on student construction of natural deduction proofs. While his students said they found the work difficult at first, they eventually all succeeded, and, to his amazement, they then learned the geometry so much faster that they still finished the book a month early.

Using Disproof by Counterexample: In any course that asks students to write proofs, one can start by giving students statements to identify as true or false, asking them to justify a determination of true "as best as you can" and to support an answer of false by providing a counterexample. One reason for beginning in this way is that most students find it easier to understand and construct disproofs by counterexample than to understand and construct even simple direct proofs. A second reason is that the more experience students have in seeing that a single counterexample disproves a universal statement, the more likely they are to understand that a general argument is needed to show that no counterexample exists. Finally, offering students mathematical statements whose truth or falsity they have to determine themselves helps make the point that proof and counterexample are first and foremost problem-solving tools.

Direct Proof: Identifying the Starting Point and Conclusion to Be Shown: The most important initial point to communicate to beginning students about proving a universal statement is that they will have to move from something that is supposed to be true to something that must be shown to follow. It then becomes natural to

- identify *what* is supposed and *what* is to be shown, which determines the outer structure of the proof, and
- address the crucial question "How do I show that?" which determines the proof's inner structure and depends critically on the definitions of the terms in the statement.

The most common type of mathematical statement is universal and conditional, having the form

For all elements x in a certain set, if <*hypothesis*> then <*conclusion*>.

A direct proof of such a statement has the following outline:

- Suppose that x is a particular but arbitrarily chosen (or "generic") element of the set for which the hypothesis is true.
- We must show that x also makes the conclusion true.

The amazing thing about this proof technique is that merely by reasoning about a single element *x*, one deduces that the conclusion follows from the hypothesis for *every* element of the set – which is typically of infinite size. The validity of the reasoning is determined by the fact that *x* is arbitrarily chosen, or "generic," which means that it has all the characteristics and only those characteristics common to every other element of the set. Hence everything one deduces about it is equally true of every other element of the set, and thus a descriptive name for this type of reasoning is ***generalizing from the generic particular***.

A dramatic way to emphasize the power of this proof method is to show how one can use it to structure proofs involving terms one does not even understand. For instance, given the statement "For all toths *T*, if *T* has a rath, then every wade of *T* is brillig," the starting point and conclusion to be shown for a proof would be "Suppose *T* is any toth that has a rath. We must show that every wade of *T* is brillig." This transformation may seem obvious to a mathematician, but it does not come naturally to many students. Yet as students venture further and further into realms of mathematical abstraction, instinctive ability to use the transformation becomes increasingly essential to their success.

Recognizing the "Suppose" and "To Show" in Proof by Contraposition and Proof by Contradiction: Once one has introduced proof by contraposition and proof by contradiction as well as direct proof, one can help students understand the differences among them by pointing out that while for each method there is something supposed and something to be shown, these "somethings" are dramatically different in each case. In a direct proof one supposes one has a particular but arbitrarily chosen object that satisfies the hypothesis, and one shows that this object satisfies the conclusion. In a proof by contraposition one supposes one has a particular but arbitrarily chosen object for which the conclusion is false, and one shows that for this object the hypothesis is also false. In a proof by contradiction one supposes that the entire statement to be proved is false, and one shows that this supposition leads to a contradiction.

Use of Definitions: Mathematically speaking, the most important part of a statement's proof is how one gets from the hypothesis to the conclusion. For most of the proofs undergraduate students are asked to construct, the majority of this task is achieved through a logico-linguistic analysis of definitions. The reason is that the inner structure of a straightforward, or routine, mathematical proof is largely determined by the meanings of the terms.

Note that, although they are frequently stated less formally, definitions are actually bidirectional. For instance, for *n* to be an even integer means that "*n* is even if, and only if, *n* equals twice some integer." Thus if we know that *n* is even, we can deduce that *n* equals twice some integer (from the "only if" part of the definition), and if we know that *n* equals twice some integer, we can deduce that *n* is even (from the "if" part of the definition).

To answer the question "How do I show that the conclusion follows from the hypothesis?" the prover needs an operational understanding of the "if" direction of the definitions of the mathematical terms in the conclusion. For example, to derive the conclusion that a certain quantity is rational, one needs to show that it can be expressed as a ratio of integers with a nonzero denominator. To derive the conclusion that one set is a subset of another, one needs to show that any element in the one set is an element in the other. To derive the conclusion that a function f is one-to-one, one needs to show that given any elements x_1 and x_2 in the domain for which $f(x_1) = f(x_2)$, one can conclude that $x_1 = x_2$. Similarly, to work forward from the hypothesis toward the conclusion, the prover needs an operational understanding of the "only if" direction of the mathematical terms in the hypothesis. Helping students translate the formal wording of a definition into such operational terms is one of the most important tasks facing a teacher in a course introducing students to proof.

One way to help students learn to use definitions is to try to induce them to see a definition as providing a test that has to be passed to decide whether something is the case. As soon as a new definition is introduced, one can introduce a range of examples, phrasing each as a question. For instance, immediately after defining rational, one can write "Is 0.873 rational?" and simultaneously ask the question out loud. To a student's answer of "yes," one can write "Yes, because" and look expectantly at the student. The student may be surprised that additional words seem to be called for but is generally able to supply the reason without difficulty (or other students may help out). One can move on to slightly more complicated examples (Is –(5/3) rational? Is 0 rational? Is 0.25252525… rational?), each time acting as if it is taken for granted that the student answering the question will give a reason. Soon students learn to give the reference to the definition without prompting, and gradually they come to understand the value of using the definition to answer such questions. Coming to see a definition as the ultimate test that determines whether or not a given object has a given property can help students accept certain facts, such as that 0 is an even number, which, surprisingly, they often disbelieve.

It is also useful to discuss alternative but logically equivalent ways to phrase definitions because it is often the case that the truth or falsity of a mathematical statement is more apparent if one uses one phrasing of a definition rather than another. Moore **[15]** gives several examples of student failure resulting from a lack of awareness of alternative versions of definitions. In an introductory course, an instructor needs to build in exploration of such alternative phrasings, reviewing the fact of their logical equivalence and showing how to operate with each version in the circumstances where it is superior to the others. Selden and Selden **[19]** argue that students' difficulty "packing and unpacking" the logic of mathematical definitions and theorems seriously undermines their ability to judge the correctness of mathematical arguments and to formulate arguments of their own. My experience supports this view. It is the main reason I give students practice in translating back and forth from formal mathematical statements to their many different informal versions. Because so many students find this difficult, I often continue to assign translation exercises throughout a large portion of the course.

Another reason to discuss alternative wordings for definitions is to compensate for the fact that quite a few students are still in the process of developing a more sophisticated concept of variable. For example, one way to state the definition of even is "n is even if, and only if, there is an integer k such that $n = 2k$," and in the usual development of many proofs it is important to be able to use this formulation. However, students with a naïve understanding of variables and quantification often make mistakes when they use it. For instance, to prove that the sum of any two even integers is even, they represent both as $2k$, thereby only considering the case where the integers are the same. To help them come to a more mature understanding of the definition, it is helpful (1) to restate it less formally (as was done previously in this discussion) without using an additional variable such as k, and (2) to write it several times using a variable but each time with a different symbol to represent it, pointing out that it is the existence of the integer k, not the symbol used for it, that is important.

In **[21]** Tall and Vinner introduced the notion of "concept image," which shed considerable light on students' understanding of mathematical definitions. A concept image for a definition is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." An overly narrow concept image leads to mistaken assumptions and may result in incorrect mathematical arguments. For students to develop concept images adequate to help them effectively evaluate abstract mathematical statements, they need experience with a broad range of examples for each newly defined term. They also need to become acquainted with the diagrams and other visual representations that mathematicians use in reasoning about the term. These might be arrow diagrams for relations and functions, the image of a nonspecific real number and its floor sitting on a number line for the study of the floor function, or a kind of blurry generic fraction with an indeterminate numerator and denominator for discussions about rational numbers.

II. Guiding Students' Fledgling Efforts

No matter how much one tries to prepare students for the process of writing proofs on their own, a certain number find it very difficult. It seems that some students cannot believe that an instructor is serious about demanding coherent expression, while others simply have difficulty putting all the pieces together in a way that makes sense. To learn as complex a skill as proof construction, most students need quite a bit of individual, back-and-forth interaction with an instructor. To the extent that one cannot act as a private tutor to every student, one can try to devise effective substitutes. For example, one can

- have students complete a few fill-in-the-blank proofs as homework to give them an out-of-class experience of participating in the development of a complete proof without making them responsible for its entire construction;
- supply a variety of model solutions for some of the homework problems to show students that their individual work is really supposed to resemble the kind of proofs that have been developed in class;
- suggest that students read their proofs out loud to test whether they are written in coherent sentences;
- discuss the kinds of errors often made in writing proofs.

Additional strategies are discussed in greater detail below.

Student Critiques of Proofs: A number of textbooks for "bridge" and discrete mathematics courses contain exercises asking students to determine whether a proposed proof for a given statement is valid or not. Campbell and Baker **[4]** developed activities that take these exercises one step further. Each activity "consists of a given statement and several different proposed proofs of that statement," some of which are valid and some of which are not. Students are divided into groups, and each group is given "one of the statement's proposed proofs, with directions…to determine if the proof is an acceptable argument," and, if so, to answer the following questions:

1) "What type of logical argument did the author use (direct, contradiction, contrapositive)?
2) How well written is the proof?
3) Was it easy to follow? Why or why not?
4) Can you think of some specific details which would make it clearer? If so, what are they?"

If students determine that the proposed proof is not an acceptable argument, they are asked to "identify all the major problems" they find with it. Each time a group finishes evaluating one proposed proof, it is given another, until each group has critiqued the entire collection. In the next class period, the students and the instructor discuss the various groups' critiques, "both on the level of identifying major issues, as well as minor problems such as style and clarity." Campbell reports that "having a variety of proposed proofs, all of the same statement, seems not only to help the students in recognizing certain logical errors, but also in developing a language of their own, recognizing that a statement can be correctly proven in a variety of ways, and learning the importance of reviewing one's work with a careful and objective eye." She also comments that students have benefited by becoming aware of the importance of format and of making proofs reader friendly.

Whole-Class Proofs: One technique for increasing student involvement in the proof-development process is for a teacher to do the writing on the board but have the students supply the individual steps. Richard L. Morrow **[16]** reported that when he uses this approach, he allows each student to give only one step so that as many students participate as possible. He wrote that "everyone gets to absorb the step, including its genesis or motivation, reason and role in the proof" and stated that the process makes it so that he "can

1) demonstrate how to go to the final steps and work backwards, when getting stuck approaching the proof from the beginning,
2) knowingly allow a proof to head off in the wrong direction and ask for suggestions on what to do when we get stuck – something which is sure to happen to many students when working alone,
3) demonstrate the value of marking up a diagram before writing down the steps,
4) show the value of getting a holistic view of the situation before putting down the series of steps – the right brain is especially useful in geometry proofs,
5) watch faces and judge how well the class or individuals are doing,
6) demonstrate that proofs do not need to be perfect or elegant to work,
7) let students know that everyone (or nearly so) is in the process of learning to do these things."

Identifying the Crux of a Proof: Many of the proofs one asks students to develop depend on a single central idea. Starting the proof-development process by trying to identify it accords with Leron's **[12]** suggestion to work down from a "top-level view of the proof." For other proofs, however, one may only come to realize the essential features after plowing mechanically through its details. Coming to see the crux of a proof in this way occurs, therefore, during the part of the problem-solving process Polya refers to as "looking back."**[17]** A practiced mathematician can easily reconstruct a lengthy proof just by recollecting its essence, but students often have difficulty when told the main idea because they are still struggling to master the underlying logic of proof construction. Becoming aware that it is possible to reconstruct proofs from a few central ideas can help motivate them to develop facility with the more routine aspects of mathematical argumentation.

Using Informal Explanations: Hodgson and Morandi **[10]** report success following an idea of Mason, Burton, and Stacey **[14]** to have students first develop an informal explanation to convince a fellow student of the truth of a statement before trying to write a proof formally. Initially, the student verbalizes the explanation, using a tape recorder to refine it until a fellow student finds it convincing. Then the student writes up the explanation carefully. Only after completing these steps does the student rewrite the explanation, filling in any necessary details and using standard mathematical language. In their article, Hodgson and Morandi follow a student through the process as she develops a proof that for all integers n, $n(n+1)(n+2)$ is divisible by 6.

Student Presentations: Having students present proofs from homework assignments for the rest of the class at the board is especially effective if started in the very first class period after proofs have been assigned. It is important, however, to make sure to preserve the self-esteem of the presenters. One can thank them for being good sports when they volunteer and point out that to the extent that they make mistakes, discussion about them helps everyone in the class avoid similar errors in the future. If a student's proofs are good, the other students see that the demands made by their instructor can actually be met by one of their own kind. If a student's proofs contain mistakes or sections that are not well expressed, an instructor can ask for suggestions for improvement from the rest of the class. A ploy is to ask students to imagine they are a research team for a large company and that if they can collectively come up with a perfect answer they will get to share a sizeable bonus. After the class has finished its critique and some changes have been recorded, the instructor can take a turn, using the opportunity both to comment on significant errors that have gone undetected and also to show students the kinds of things the instructor will be looking for when grading students' work.

When I use this technique, I discuss small details as well as larger issues, but I try to put my criticisms in perspective, explaining frankly that certain corrections are more important than others, but that I also care about what might seem to be relatively minor points. For instance, if a student's proof states that a certain number, say n, is even because it equals $2k$, I would ask what was missing. Most likely, based on the emphasis I had previously placed on definitions, one of the other students would tell me to add "for some integer k." I would agree, pointing out that, for example, $1=2\times(1/2)$ and yet 1 is not even and adding that it is not enough for n to be 2 times something – that *something* has to be an integer.

My primary reason for engaging in these kinds of critiques is to provide immediate feedback on students' proof writing, but an important secondary reason is to counteract student anxiety about how their proofs will be evaluated. Since there is more than one right way to construct any given proof and since different instructors may well have different standards of correctness, I feel obliged to try to give my students a sense of the range of proof styles I consider acceptable and to indicate which parts of a proof I consider most important. So when I critique student proofs, present my own, and write proofs at the board that have been developed collaboratively with members of the class, I discuss alternative ways of expressing the steps that I would consider acceptable. I also talk about conventions of mathematical writing, such as giving only part of the reason for a certain step, enough to indicate that the writer of the proof has considered and resolved the issue but not so much as to overload the proof with unnecessary verbiage. In addition, I point out that the amount of detail included in a proof varies considerably depending on the intended audience. In my courses I generally suggest that students address their proofs to a fellow student who has missed the last few classes.

At DePaul University some instructors have begun requiring students to present solutions to selected proof problems individually during office hours. Some require students to present one proof or disproof from each of the main types discussed in the course, while others offer students the possibility of raising their grade on a test by presenting a revised version of one of the problems they missed. Students are alerted that the instructor may stop to ask for clarification and base part of their grade on how effectively they respond, but because the presentations are private students do not need to worry about being embarrassed in front of their peers. In some cases the instructor's questions simply allow the student to demonstrate understanding of the reasons for certain steps; in other cases they raise more serious issues about the correctness of the argument or the incorrect use of terminology. Because several of the student's difficulties can be cleared up in the same session, such one-on-one student-instructor interaction can result in significant improvement in student understanding.

Rewriting Proofs: Requiring students to rewrite proofs until they are correct is a useful way to help students improve their proof-writing skills. In a large class it may be impossible for an instructor to find time to provide suggestions for improvement for the majority of assigned problems, but it may be possible to make sure that students rewrite at least one of each type of proof that is assigned. Nancy L. Hagelgans, Ursinus College, gave the following concrete suggestions. **[8]**

1) Have students submit double-spaced, word-processed drafts electronically, except for first drafts taken from tests.
2) Write comments in pencil.
3) Comment on the appropriateness of the proof method or the lack of evident method.
4) If the method is appropriate, comment on the argument.
5) If the argument is valid, comment on the English composition.

6) Mention the good points: “A great first sentence!” “Clear organization!” “Good choice of method of proof!” “Excellent proof so far!”
7) Have conferences outside class with a few of the weakest students after several drafts.

Some variations she suggests are to have the whole class discuss selected first drafts that are projected on a screen, have student pairs discuss and write comments on each others’ first drafts in class, and have students write comments on copies of selected first drafts. Another variation is to have students work on proofs in groups in class and go from group to group reviewing their work and offering hints on how to correct it.

Addressing Process Issues: To help students cope with the often frustrating enterprise of mathematical discovery, one can encourage class discussion about the psychological aspects of the process. For instance, if a few students have found a counterexample for a mathematical statement that stymied a majority of the class, one can ask the successful students to share the thoughts that went through their minds when the counterexample occurred to them. One can also point out that mathematical discovery may involve emotional ups and downs, that even the best mathematicians find mistakes in their arguments which force them to abandon one approach and seek another. For example, work of Schoenfeld **[18]** supports the view that while successful problem solvers are persistent, they readily change to new approaches when previous ones do not appear to be working, though they might eventually return to a previous approach if a new attempt seems unsuccessful.[1]

To assist students in structuring their time when they are trying to determine truth or falsity of a mathematical statement, one can suggest that they begin by imagining they actually have an object or objects satisfying the conditions described in the hypothesis. They can then ask themselves whether the conclusion must necessarily follow. If, after some effort, they do not see why this must be so, they can explore the possibility that the statement might be false by trying to think of elements that satisfy the hypothesis but not the conclusion. If this effort also fails, they can posit a situation where the hypothesis is true and the conclusion is false and try to derive a contradiction. If this method also seems to lead nowhere, the very process of having tried it and the other approaches may have resulted in insights that could lead to greater success when one of the previous approaches is tried again.

III. Motivating the Need for Proof

A common use of proof is to affirm the general truth of properties that one has seen to be true in some cases, thereby coming to understand the essential reasons why the property always holds. While all introduction-to-proof courses try to convey this point, in courses where exploration and experimentation play a major role, it is the primary way the need for proof is introduced. For example, in the Mount Holyoke course Laboratory in Mathematical Experimentation **[6]** students work in groups on laboratory-style projects, most of which use the computer as an experimental tool to generate examples. Students are expected to come to see patterns and are then prompted to conjecture generalizations. Finally they are asked to support their conjectures with analytical arguments including, when possible, complete proofs. Projects are chosen from, among others, number theory, dynamical systems, and graph theory. The Franklin and Marshall College course Introduction to Higher Mathematics is structured in a similar way. **[13]** An initial “module” guides students through a sequence of increasingly pointed questions and activities to discover and verify basic properties of even and odd integers. Another module leads students to discover patterns related to the Fibonacci sequence by having them fill in values in a table for n, f_n (the nth

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[1] The Nova progam “The Proof,” which describes Andrew Wiles’ discovery of a proof for Fermat’s Last Theorem, provides a powerful example for the effectiveness of this strategy.

Fibonacci number), and $S(n)$ (the sum of the first n Fibonacci numbers). Later modules treat a variety of mathematical topics, such as polynomials and complex numbers, combinatorics and graph theory, difference equations and iteration, and number theory. The Foundations of Computing course at Butler University, developed by Peter Henderson, **[9]** incorporates a once-a-week laboratory for exploration and experimentation alongside a more conventionally organized exposition of discrete mathematics that emphasizes logic and proof. In the lab sessions, students grapple with theoretical problems, which are often phrased engagingly as puzzles. These motivate the need for proof by requiring ingenuity and proof-like analysis (such as recursive thinking and identification of invariant properties) to solve.

When students are skeptical about the need for proof, a particularly effective way to motivate it is to have them evaluate statements about whose truth or falsity reasonable people might reasonably disagree. Fortunately, there are more such statements than one might think because what is obvious to a mathematician is not always obvious to a student. It is also possible to find relatively elementary statements upon which most people would need to reflect in order to reach a definitive answer. Such statements are especially effective when used for student presentations in class. For instance, consider the statement "For all integers a, b, and c, if a divides bc then a divides b or a divides c." If one assigns a homework problem asking students either to prove or provide a counterexample for this statement and then uses it for class discussion, it is common for one part of a class to claim it is false and another to say it is true. Once when two students from each group were chosen to go to the board to present their solutions, the result was one false proof, one partial "proof," one incorrect counterexample, and one correct counterexample. Such an outcome is a powerful argument for the importance of careful reasoning, especially if one points out that the ability to come up with correct answers to such mathematical questions provides the theoretical foundation to be able to engineer airplanes that do not crash, develop encryption systems to keep transmission of credit card information secure, and so forth.

This approach was developed as a formal teaching method, called "scientific debate," by a group of mathematics educators in France. In a first step, "the teacher initiates and organizes the production of scientific [mathematical] statements by the students. These are written on the blackboard without any immediate evaluation of their validity." In the second step, "the statements are put to the students for consideration and discussion. They come to a decision about their validity by taking a vote, with each opinion supported in some way, e.g. by scientific argument, by proof, by refutation, by counter-example, etc." In the third step, "the statements which can be validated by a full demonstration become theorems, whilst those which are established as incorrect are preserved as "false-statements," with a corresponding counter-example."**[1]**

The approach is taken even further in courses that use the "Moore method" or a "modified Moore method." In these courses students are given a list of definitions and an ordered set of statements proposed as possible theorems. They are given the job of proving the statements that are true and finding counterexamples for those that are false but are not allowed to consult textbooks or obtain solutions from an outside source. Classes consist primarily of presentations by students of their work, which is followed-up by questions and comments from members of the class. The method, originated by R. L. Moore for graduate courses at the University of Texas, has been modified by others to adapt it for use with a broader range of students and in less advanced courses. For instance, Chalice **[5]** includes elementary exercises on definitions to help the average student understand how they are applied to simple examples, and he encourages students to visit during office hours for hints on problems that give them difficulty. He also gives three exams during the semester, and when students are preparing for an exam he makes available to them careful proofs of the theorems that will be covered.

Conclusion

A few years ago I had an experience with one particular class that made a special impression on me. The class was unusually small, only twelve students, and was the second quarter of a sequence. The previous quarter had dealt with logic, an introduction to direct and indirect proof, mathematical induction, and elementary combinatorics, all interwoven with various computer science applications. The second quarter was to cover set theory, function properties, recursion, some analysis of algorithms, relations on sets, and an introduction to graph theory, also with an admixture of applications. The class met only once a week but in three-hour sessions.

The small size of the class and the length of the sessions gave me a chance to work with students more intimately than usual. I began each period by having students discuss in groups of three or four the homework they had prepared for that day and went from group to group talking with each at length. Overall the class atmosphere was excellent, and several students showed the kind of eager, enthusiastic intelligence that is a teacher's joy. What surprised me was that as the course moved from one topic to the next, almost all the students who had attained a relatively sophisticated level of achievement in dealing with a previous topic made it clear that they felt they had to struggle to succeed with the next. Yet as we worked through their questions and difficulties, they ultimately performed excellently with the new topic as well. Their understanding of general methodological principles clearly made it easier for them to learn the new material but it did not make it trivial for them.

This experience brought home to me more effectively than any before that abstract mathematical thinking is not something that either one is able to do or one is not able to do. Because of the experience I have become especially conscious of the need to respect my students and never to act surprised by their questions. Even when a student asks a question whose answer I have already discussed, I try to respond to it as if it were fresh. After all, nobody can concentrate 100% of the time when new ideas are coming in fast and furiously. In all likelihood the student was not mentally prepared to absorb the answer when I previously addressed the question. For the student to formulate the question means that they have thought about the issue, want to know the answer, and are probably ready to understand it. That is cause to celebrate. It may also be that clarifying the issue at this point in the course (if possible in a slightly different way from that presented earlier) will give the other students in the class greater insight also.

My main advice to those teaching courses whose goal is to develop students' mathematical reasoning powers is to play an activist role but recognize that achieving success is a long-term process. I have sometimes been surprised when students who in my view fell far short of achieving the levels of accomplishment I strive for tell me how valuable they found the course in helping them do better work in their other courses or (I am always pleased to hear) in their jobs.

The analogy I like to draw is of a child learning to walk. It takes months of daily effort for most children to take their first steps and several more months until they actually become steady on their feet. When a child is trying to move from one stage to the next in learning to walk and has failed several times, we don't say “Forget it.” We remain calm, good humored, and encouraging. And when the child finally succeeds, we spare nothing in expressing our delight.

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