

The Language of Quantification in Mathematics Instruction

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As a relatively young college professor in 1979, I first began teaching mathematics courses whose primary aim was to help students improve their ability to assess truth and falsity of mathematical statements. At the outset, I had a lot of good intentions and much personal experience doing mathematics myself, but I had very little background in the formal aspects of mathematical thought. It soon became clear that the unconscious instincts for logic and language which had enabled me to succeed were not shared by the large majority of my students.[4] And my lack of knowledge about the formal basis for those instincts made it difficult for me to communicate with them to help correct their errors and build their capacity for abstract reasoning.

As I attempted to gain insight into my students' problems, I consulted the work of cognitive psychologists and researchers in mathematics education and discovered that the kinds of mental processing problems my students exhibited are extremely common in the population at large. This discovery helped reconcile me to the existence of these problems in my students and inspired me to try to invent remedies. My efforts were aided by continual interaction with my students, by a few of my own experiments, and by acquiring some technical knowledge about logic and language. My approach was also informed by a growing sensitivity to the use of language outside of mathematics and the realization that some of my students' problems may have resulted from confusion between the way certain words are used in everyday speech and their more technical meanings in mathematics.

When I began incorporating instruction in the logic and language of mathematics as a theme running through my courses, student performance improved significantly. But I was struck by how much effort was required for many of them to overcome the linguistic habits that were leading them into mathematical error. Since the ability to acquire language is greatest in the young, cultivating students' sensitivity to logic and language in the pre-college years could check the formation of these habits before they become deeply ingrained. The success of programs such as TexPREP, run by Manuel Berriozabal in San Antonio, provides objective support for such a conclusion.[2]

Some Formalism. At the heart of mathematical discourse are words referring to quantity: 'all' and 'some,' along with variations such as 'every,' 'any,' and 'no.' Almost all important mathematical facts contain at least one of these words. For example, consider the list of statements in Table 1.

1. For all pairs of real numbers a and b , $a + b = b + a$.
2. In every right triangle, the square of the hypotenuse equals the sum of the squares of the other two sides.
3. No integers a and b have the property that $\sqrt{2} = a/b$.
4. Given any integer, there is some integer that is larger.
5. Some positive integer is less than or equal to all positive integers.
6. For all real numbers $\varepsilon > 0$, there is some integer N such that for all integers $n > N$, $|a_n - L| < \varepsilon$.

Table 1

In order to work effectively with such statements, students need to have a sense for their logical form. For instance, how can we figure out if a given *all* statement is true? What does it mean for an *all* statement to be false? What are other, equivalent ways of expressing an *all* statement? If a given *all* statement is true, what can be deduced?

A *universal statement* is one that can be written in the form 'All A are B .' To deal capably with such statements, one needs to be able to express them in a variety of equivalent ways. For instance, 'All squares are rectangles' can also be stated as 'Every square is a rectangle,' 'Any square is a rectangle,' 'For all squares x , x is a rectangle,' 'If something is a square, then it is a rectangle,' and 'For all x if x is a square, then x is a rectangle.' The fact that universal statements can be expressed as *if-then* statements is of particular importance in mathematics.

An *existential statement* is one that can be expressed as 'Some A are B .' Such statements also have alternate, equivalent formulations. For instance, 'Some rational numbers are integers' can also be written, 'There is at least one rational number that is an integer,' 'For some rational number, that number is an integer,' or 'There exists a rational number x such that x is an integer.'

A beautiful duality governs the truth and falsity of universal and existential statements. To say 'It is false that all A are B ' is equivalent to saying 'Some A are not- B ,' or, 'Some A have the property that they are not B .' And to say 'It is false that some A are B ' is equivalent to saying that 'All A are not- B ,' or, 'All A have the property that they are not B .' For example, the statement 'It is false that all people are honest' means that 'Some people are dishonest,' and the statement 'It is false that some doors are locked' means that 'All doors are unlocked.'

In general, to establish the truth of a statement of the form 'All A are B ,' we suppose we have a particular but arbitrarily chosen (or generic) object, that is an A . Let's call this object x . We then show that x is a B . This method of reasoning is sometimes called *generalizing from the generic particular* because since we make no special assumptions about x that do not apply equally to every other A , everything we deduce about x applies equally to every other A . For instance, to show that the square of any odd integer is odd, we suppose x is any particular but arbitrarily chosen odd integer. By definition of odd, $x = 2k + 1$ for some integer k . It follows that $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd (because it can be written in the form $2 \cdot (\text{an integer}) + 1$).

What Research Shows. The ability to rephrase statements in alternate, equivalent ways, to recognize that other attractive-looking reformulations are not equivalent, and to have a feeling for truth and falsity of universal and existential statements are crucial mathematical problem-solving tools. Yet numerous studies show that students do not acquire these abilities spontaneously.

For instance, there is much evidence that a majority of people perceive the statements 'If A then B ' and 'If B then A ' as equivalent and do not readily deduce that 'If A then B ' implies 'If not B then not A .' [1, pp. 292-297] And I've found that approximately 70% of large samples of students select 'No mathematicians wear glasses' as the sentence that "*exactly* expresses what it means for the sentence 'All mathematicians wear glasses'" to be false, whereas only about 20% choose the correct negation 'Some mathematicians do not wear glasses.' [4] Recently, Dubinsky and Yiparaki have collected evidence showing that a significant fraction of students interpret a statement of the form 'There is a ___ such that for all ___, ___' to mean the same as 'For all ___, there is a ___ such that ___' [3] And both Epp [4,5] and Selden and Selden [9,10] have documented some of the ways in which students' inability to "unpack the logic of mathematical statements" impairs their ability to understand and construct mathematical arguments.

Table 2 illustrates how some of these misconceptions manifest themselves inside the mathematics classroom.

1.	Just because it's true that All irrational numbers have nonterminating decimal expansions.	it doesn't follow that All numbers with nonterminating decimal expansions are irrational.
2.	For all positive numbers a , there is a positive number b such that $b < a$.	There is a positive number b such that for all positive numbers a , $b < a$.
3.	Just because it's false that All rational numbers are integers.	it doesn't follow that No rational numbers are integers.
4.	Because it's true that For all positive numbers x , if $x^2 < x$ then $x < 1$.	it does follow that For all positive numbers x , if $x \neq 1$ then $x^2 \neq 1$.

Table 2

Confusion Between Mathematical and Everyday Language. One reason students may have problems using logic correctly in mathematics and other technical situations is that in informal settings certain forms of statements are often interpreted in ways that differ from their formal meanings. Here is one example.

Imagine that a teacher promises a class: "All those who sit quietly during the test may go out and play afterwards," and imagine also that this teacher then allows noisy students to go outside and play along with the students who were quiet. From the point of view of formal logic, the teacher's actions are perfectly consistent with the teacher's words. After all, the teacher only said what would happen to students who sat quietly during the test; nothing was said about students who made noise. Most observers of the scene, however, would judge such a teacher lacking in resolve. Presumably, the teacher would not have made the statement without intending students to infer the converse: "All those allowed to go out and play after the test will have sat quietly during it," or, equivalently, "All those who do not sit quietly during the test may not go out and play afterwards."

Even mathematicians sometimes take advantage of people's perception that 'All A are B ' and 'All B are A ' are equivalent. For example, a text might define even integer by stating: "any integer with a factor of 2 is *even*," (or, equivalently, "an integer is *even* if it has a factor of 2"). Yet later in showing why, say, the sum of two even integers is even, the text would assume the truth of the converse statement, "any integer that is even has a factor of 2" (or, "if an integer is even then it has a factor of 2").

The problem is, of course, that situations arise regularly in mathematics and other technical fields where 'All A are B ' is true whereas 'All B are A ' is false. Item 1 of Table 2 is one such case. So despite the many times we can get away with being imprecise about the logical relationship between the two types of statements, in order to be really successful, we need to develop awareness that it is possible for the statements to have different truth values and even to be on the lookout for situations where this occurs.

The example cited above is just one of many. Recently, Wells has made available on the Internet an entire lexicon of grammatical usages whose everyday meanings differ somewhat from their mathematical ones.[11]

Suggestions for Mathematics Instruction

A. As a result of the 1989 NCTM *Standards* [8], many teachers are now asking students to explain their reasoning or justify their answers to various problems. Critiquing students' work, though time-consuming, gives teachers an excellent opportunity to point out fallacies in a student's own reasoning or to show that more evidence is needed to adequately support a student's conclusion. When given by teachers with a solid command of mathematical language and logic, such feedback can be of enormous benefit to students' intellectual development.

B. Logical puzzles are both fun and almost unbeatable at helping students learn that deductive reasoning consists of chains of inferences. They are also a marvelous introduction to the concept of reasoning by contradiction because to solve many of them, one shows that certain possibilities lead to contradictions and can therefore be eliminated. Logical puzzles suitable for all K-12 grades are widely available. If every math teacher devoted just a few hours a year to having students work such puzzles, a marked improvement in students' reasoning abilities might result.

C. Even in the earliest grades, students can be given activities to increase their sensitivity to the language of quantification. Consider, for instance, showing them the following picture:



One could ask whether each of the following statements is true or false:

1. All the white objects are squares.
2. All the square objects are white.
3. No square objects are white.
4. There is a white object that is larger than every gray object.
5. Every gray object has a black object next to it.
6. There is a black object that has all the gray objects next to it.
7. All objects that are not small are not gray.

Observe that questions 1-7 address all the logical difficulties illustrated in Table 2. Perhaps if students became accustomed at an early age to dealing with such questions in simple situations, they might develop an feeling for the deep grammar of mathematics which would lead them, as more mature students, to avoid the kinds of misunderstandings shown in Table 2. A recent study indicates that the way peoples' brains process the deep grammar of a language differs depending on whether they developed fluency in the language in infancy or at about age 11.[5]

D. As soon as letters have been introduced to stand for unknown or unspecified quantities, teachers can ask questions involving variables. The following two problems are taken from the Russian mathematical series translated as part of the University of Chicago School Mathematics Project [7]:

1. (Grade 3, age 9) For what values of the letters are the following equalities true?

(a) $36 \cdot b = b$	(b) $10 \cdot c = 10$	(c) $12 \cdot a = a \cdot 12$	(d) $a \cdot a = a$
(e) $49 \cdot a = 0$	(f) $c + c = c$	(g) $b + b = b + 6$	(h) $b \cdot 0 = 0$
2. (Grade 5, age 11) It is given that x and y are positive numbers, $x < y$. Replace each asterisk with the symbols $<$, $>$, or $=$ so as to obtain a true equality or inequality.

(a) $0 * x$	(b) $-y * 0$	(c) $-x * y$	(d) $y * -x$
(e) $ x * -x$	(f) $ y * y$	(g) $-x * x $	(h) $ -x * -y $

Note that for each of these problems students are only asked to give an answer not to justify it. But in order to answer the questions correctly, students need to imagine substituting various values for the variables and testing the resulting expressions for truth and falsity. This is an excellent way for them to start thinking about the issue of truth and falsity of general mathematical statements. Even if such special questions are not added to the curriculum, teachers can improve student understanding by asking them to

"find all numbers x that make this equation true" rather than always phrasing the request as "solve this equation for x ," which many students come to see as a purely mechanical procedure involving the mysterious " x ."

E. As early as pre-algebra, one can introduce the idea of disproof by counterexample. One could start by using the kind of exercise suggested in item C above. Or one might invite the class to discuss whether a statement like "All eight-graders are lazy" is true or false. Because of their personal interest in this statement, it should not be hard to get general agreement that the existence of just one non-lazy eighth-grader is enough to falsify it. One could then go on to ask:

Sam claims the equation $(a + b)^2 = a^2 + b^2$ is true for all values of a and b . Ann disagrees. Who is right? Justify your answer.

F. At about the same time as one introduces disproof, one can have students justify statements by simply citing a universal property such as the commutative, associative, or distributive law or one of the laws of exponents. For instance one could pose the following question:

Dan claims the equation $k(k + 5) = k^2 + 5k$ is true for all values of k . Jan disagrees. Who is right? Justify your answer.

Or, more challengingly (from a step in a mathematical induction problem),

Dan claims the equation $k2^{k+2} + (k + 2)2^{k+2} = [k + (k + 2)]2^{k+2}$ is true for all values of k . Jan disagrees. Who is right? Justify your answer.

Of course, the hoped for answer in both cases is something like "Dan is right. This equation is true for all values of k because of the distributive law." As one asks about more and more complicated expressions, students develop an increased mastery and appreciation for the power of universal properties.

G. Stating universal properties in words as well as symbolically also enhances students' ability to apply them in a broad range of situations. For instance, one can express one of the distributive laws by saying that "one number times the sum of two others equals the sum of the first times the second plus the first times the third," while pointing to the various parts of the expression as one refers to them. This seems to have a positive effect on students who get stuck identifying the law with the particular letters used to symbolize it. If, for example, we express this distributive law using the formula $a(b + c) = ab + ac$, many students have difficulty applying it to the expression $b(a + c)$.

H. Once students have developed a little more experience with algebra and the concept of truth and falsity of mathematical statements, one can introduce the notion of proof proper, asking students to tackle a question like the following:

Is the equation $a^2 + b^2 = -1$ true for all values of a and b , for some values of a and b but not for others, or for no values of a and b ? Justify your answer.

Assuming that students answer the question correctly, the main problem they have in justifying it is realizing that citing a few examples is not sufficient to establish its truth. However, previous experience thinking about properties that are true for some values of the variables and false for other values virtually forces them to agree that the fact that a property holds in a certain number of cases does not imply it must hold in all cases.

Of course, the justification we would like students to give is something like "Since the square of any real number is never negative and the sum of any two numbers that aren't negative is not negative, it is impossible for the sum $a^2 + b^2$ to be negative regardless of the choice of a and b ." Students who answer correctly and give a few examples as support can be complimented for seeing part of the picture, but it should be made clear that their justification is incomplete.

I. At about the same time as one introduces the concept of proof, one can pose problems to help resolve the kinds of common misconceptions listed in Table 2. For example, one can ask students the following questions, giving them feedback either individually or in group or class discussions.

1. Given that all irrational numbers have nonterminating decimal expansions, does it follow that all numbers with nonterminating decimal expansions are irrational? Explain your answer.
2. Consider the set $S = \{-2, -1, 1, 2\}$. For each statement below, determine whether it is true or false. Explain your answers.
 - (a) For all x in S , there exists a y in S such that $x + y = 0$.
 - (b) There exists a y in S such that for all x in S , $x + y = 0$.
3. (a) Show it is false that all rational numbers are integers.
 (b) Is it true that no rational numbers are integers? Explain your answer.
4. The following statement is true: For all positive numbers x , if $x^2 < x$ then $x < 1$.
 - (a) Suppose x is a positive number for which $x < 1$. Could x^2 be less than x ? Explain your answer.
 - (b) Is the following statement true or false? For all positive numbers x , if $x < 1$ then $x^2 < 1$.

Exercise 2 gives students a relatively simple environment in which to explore the truth and falsity of statements with the same syntax as those in item 2 of Table 2. And exercise 4 probes the relation between a conditional statement and its contrapositive.

J. Clever teachers at all levels of instruction have devised methods for enabling students to obtain correct answers to problems in ways that circumvent their difficulties with logic and language. The problem is that using these methods deprives students of the opportunity to improve those aspects of their reasoning skills which more traditional methods require. Epp gives several examples of this phenomenon: solving inequalities using test points rather than the logic of *and* and *or*; using the vertical line test without an adequate tie-in to the definition of function; and finding the inverse of a function without explicitly using the definition of inverse.[4] Another example is the presentation of polynomial multiplication. Because it would take more time to teach students to appreciate the power and make effective use of the universal rule for multiplying polynomials (multiply each term of one by each term of the other), many teachers just present FOIL, which is effective only for finding the product of two binomials. The problem is that not only does use of FOIL result in college students' not being able to multiply general polynomials, but more importantly use of this method fails to give students the chance to broaden their understanding about how to apply universal statements in general.

K. A number of bright, capable students become disillusioned with mathematics because the rules they are taught seem so mysterious. It is understandable that as teachers we don't want to alienate our students by burdening them with explanations they can't understand. But we may have gone overboard and may now be underexplaining. Even when students don't follow every detail, many are reassured to hear reasons for rules that would otherwise seem arbitrary. So, for instance, if students ask why $1/0$ is undefined, we can say it is because in general a/b is that number which when multiplied by b gives a . So since there is no number which when multiplied by 0 gives 1, $1/0$ is not a number. Similarly, if students ask why $-(-1) = 1$, we can say it is because in general $-a$ is that number which when added to

a gives 0, and then we can ask what number when added to -1 gives 0. The answer, of course, is 1. So $-(-1) = 1$. Note that a sentence like " a/b is that number which when multiplied by b gives a " is linguistically complex but mathematically important because every inverse relationship is defined by a sentence with a similar grammatical structure.

Benefits for Students' General Intellectual Development. The linguistic framework used for mathematics consists of a small set of terms and rules. Most of these are part of the language we use everyday, but everyday language contains additional rules and conventions which are occasionally at odds with those used in mathematics. In ordinary discourse, we use context to resolve ambiguity. Indeed, a small number of children appear to be able to use this method to sort out differences between ordinary and mathematical usages without apparent assistance from their mathematics teachers. Whatever the reason for the success of this small number, the ability of the large majority of children to make effective use of the language of mathematics could be improved through explicit instruction in their mathematics classes. Surely one of the most valuable roles schools can play in society is to level the playing field for students from different backgrounds and abilities.

The precision of thought and clear deductive reasoning which has traditionally been associated with mathematical thinking has value far beyond the classroom. The same deductive logic used in careful mathematical reasoning is used to tackle tough problems in all fields. As mathematics teachers we should strive to help our students improve their abilities not only to come up with correct answers to traditional mathematical problems (though that is important) but also to use the language and logic of mathematics in ways that will potentially transfer to benefit their performance in many different areas of their lives. This may well be the most significant contribution we can make to society.

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