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A Unified Framework for Proof and Disproof

To a person with advanced mathematical training, the following proof seems almost like common sense.

THEOREM A. *The square of any odd integer is odd.*

PROOF. Suppose that x is any odd integer. Then $x = 2k + 1$ for some integer k, and so

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is odd.

For many years I have been teaching college students the kind of abstract mathematical reasoning embodied, in simple form, in the foregoing proof. Such reasoning is important not only for students majoring in mathematics but also for those concentrating in computer science, electrical engineering, economics, and other disciplines.

Although much of what I teach is best left at the college level, incorporating certain parts into the high school curriculum is desirable. One reason to do so is that college teachers often assume that students with three or four years of high school mathematics can read and write proofs without any special, extra instruction. Yet for large numbers of college students, even a proof as simple as the one shown contains many mysteries. Here are typical questions that students raise:

The majority of students lack intuitive acceptance of a few basic principles of logic

1. I have checked that 3^2, 5^2, 7^2, 9^2, and 11^2 are odd. Isn't that enough?
2. All of a sudden, this variable x appears. Where does it come from?
3. Why does the proof begin by supposing something?
4. What prompts all those algebra steps? How would I ever know to do them?
5. Why does the proof end where it does?

FILLING IN THE GAPS: QUESTION 1

The hardest part about teaching proof is deciding how much to tell. For that elite group of students who simply accept the validity of the foregoing proof and have no difficulty constructing similar ones, the teacher need do nothing beyond making an occasional offhand remark. But for the majority of students, who are genuinely puzzled and ask questions like those just listed, something more must be offered. Several decades of interacting with such students has convinced me that the main factor differentiating them from the elite group is their lack of acceptance, at an intuitive level, of a few basic principles of logic. So in my own classes and in the book that I developed from my experiences (Epp 1995), I try to fill the gaps in their understanding so that the answers to questions 1–5 arise naturally.

My aim is always to seek common ground between students' own experiences and the logical principles that I am trying to convey. As a result, although I find it convenient to explain some principles by using the notation of symbolic logic, I focus mostly on English because that is the language the students will use when they analyze mathematical statements and develop their own mathematical arguments. And because the acceptance of the truth and falsity of general forms of statements is often context dependent, I try to introduce each new logical principle with statements whose truth and falsity students can appreciate without difficulty.

In a class of college students, for instance, I might pave the way for a formal definition of the truth and falsity of a universal—"for all"—statement by asking whether it is true or false that every student in the class is more than twenty years old. Normally, one person will say that he knows that it is false, and I will ask—with a smile, because it is "obvious" to all—how he can be so sure. He plays along with my act and says that it is false because he himself is not over twenty. Ah! So if just one student in the class is not over twenty, then the general statement that "Every student in the class is over twenty" is false? Yes, that is right. Even if some other students *are* over twenty? Yes, that is right.

Okay, but what about the statement "Every student in the class is over fifteen years old"? Is it true

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or false? True? How can we be sure? Take a poll of the class to verify that each individual student is over fifteen. Ah! So for us to be able to claim correctly that a certain property holds for all members of some set, the property must hold for every member of the set individually. Yes, that is right.

For question 1, then, why is it not enough to check the truth of the property in the opening theorem for just a few odd integers? It is not enough because the theorem claims that a certain property (having an odd square) is true for *any* odd integer. If, somewhere in the infinity of all odd integers, just one odd integer exists—perhaps so large that no human being has even written it down!—whose square is not odd, the theorem would be false. So we have to know that this result cannot happen to be "mathematically certain" that the theorem is true.

FILLING IN THE GAPS: QUESTION 2

The great advantage of using variables in mathematics is that they allow us to give temporary names to objects so that we can refer to them unambiguously while performing all manner of complicated computations and deductive-reasoning steps that involve them. The French mathematician Jean Dieudonné once referred to the "boldness" of giving a name to an unknown quantity and then working with it "as if it were a known quantity" (1972, 102). And this approach *is* bold. It is also among the most useful techniques for solving mathematical problems.

From the very beginning of their study of algebra, we teach students how to use variables to stand for unknown quantities in equations and how to manipulate expressions involving variables. To give students practice in using variables in the sentences that are used in deductive reasoning, I might pick up on the interaction discussed in the previous section by asking the class to state some different ways to express the idea that "Every student in the class is over fifteen." Typical responses would be "All students in the class are over fifteen" and "Each student in the class is over fifteen." Fine, but in mathematics we often do things a little strangely! What about this sentence? "Given any student, if the student is in this class, then that student is over fifteen." Does this sentence have the same meaning as the others? Well, yes, although it sounds somewhat peculiar. How about these sentences? "For any student x, if x is in this class, then x is over fifteen" or "For all students x, if x is in this class, then x is over fifteen." Also peculiar, but yes, these sentences have the same meaning as the first one.

At this point, having discussed ordinary conditional statements earlier in the course, I would define a *universal conditional statement* to be one of the form "For all x in D, if $H(x)$, then $C(x)$," where x is a variable taking values in a set D and where $H(x)$ and $C(x)$ are properties that may or may not be true for a given x. Property $H(x)$ is called the *hypothesis;* and $C(x)$, the *conclusion* of the statement.

For instance, D could be the set of all students; $H(x)$, the property "x is in this class"; and $C(x)$, the property "x is over twenty" or "x is over fifteen." As we saw in the examples, for such a statement to be true, $C(x)$ must be true for each individual x in D for which $H(x)$ is true. For the statement to be false, at least one x exists in D for which $H(x)$ is true and $C(x)$ is false. Or, more formally, the negation of "For all x in D, if $H(x)$, then $C(x)$" is "There exists an x in D such that $H(x)$ is true and $C(x)$ is false."

To help students develop the ability to translate back and forth between formal and informal modes of expression, I give them such exercises as the following:

a) Rewrite the statement "For all real numbers x, if $x > 2$, then $x^2 > 4$" in various ways, without using a variable.

b) Rewrite the statement "The square of any odd integer is odd" in the form "For all integers x, if ______ then ______."

Answering part *(a)* helps students develop their ability to deal flexibly with the kind of universal statements that are ubiquitous in mathematics. And, of course, the correct response to part *(b)*, "For all integers x, if x is odd, then x^2 is odd," is exactly what is needed to understand the answer to question 2. Where does the x come from? It is just a temporary name that we give to something so that we can refer to it repeatedly without becoming confused.

It is quite bold to name an unknown quantity and then work with it as if it were known

FILLING IN THE GAPS: QUESTIONS 3 AND 5

Sometimes the truth or falsity of a statement "For all x in D, if $H(x)$, then $C(x)$" can be determined by checking $C(x)$ individually for each x in D for which $H(x)$ is true. For example, is it true or false that "For all even integers x, if x is less than 25, then x can be expressed as a sum of three or fewer perfect squares"? Since only finitely many even integers are less than 25, we can check each one to see whether it can be expressed in the specified way.

But when infinitely many elements exist in D for which $H(x)$ is true, it is impossible to check each individually. What saves the day is the logical principle known as *universal generalization* or *generalizing from the generic particular.* The great mathematician and philosopher Alfred North Whitehead was referring to this principle when he wrote, "Mathematics, as a science, commenced when first someone, probably a Greek, proved propositions about 'any' things or about 'some' things without specification of definite particular things" (1958, 7).

If they can outline a proof, they will succeed on any advanced mathematics examination

According to this principle, the following is true.

To prove a statement of the form

For all elements x in a set D, *if* $H(x)$, then $C(x)$,

(a) suppose that x is any (particular but arbitrarily chosen) element in D for which $H(x)$ is true,

and

(b) show that x makes $C(x)$ true.

The word *any* in the supposition is crucial. To say that x is *any* element of D for which $H(x)$ is true means that although we are to imagine it as one particular element, we are not to think of it as having any properties other than those possessed by every other element of D that satisfies $H(x)$. In other words, x is a *generic* element of D that satisfies $H(x)$.

When stated as a technique of proof, this deductive principle is often called the *method of direct proof.* The outline of a direct proof depends on only the form of the statement to be proved, not on its content. To help students realize the significance of this point, I ask them to state what one would suppose and what one would show to prove something nonsensical like "For all bilops x, *if* x is a gragon, then x is a trexer." The answer, of course, is that we suppose that we have any bilop x that is a gragon, and then we must show that x is a trexer.

I often promise students that if they learn how to write just the outline of a proof for a universal conditional statement, I will virtually guarantee them some success on any advanced mathematics examination that they ever take. When asked to prove such a statement, all they have to do is write what they need to suppose and what they need to show and then claim, "This is obvious." With luck, their teacher will agree, take them to task for not providing the details, and then give them partial credit.

To help students come to see proof as a process, I suggest that they put the statement to be proved in universal conditional form and then ask themselves two questions: "What do I need to suppose?" and "What must I show?" When considering theorem A, they would rewrite

The square of any odd integer is odd

as

For all integers x, if x is odd, then x^2 is odd.

Then they would ask, "What do I need to suppose?" and "What must I show?" The correct responses would be "I need to suppose that x is any integer that is odd" and "I must show that x^2 is also odd," which answer questions 3 and 5.

FILLING IN THE GAPS: QUESTION 4

Once students have assimilated the basic structure of a proof and know how to figure out what they are supposing and what they must show, it is not difficult to convince them to ask themselves, "How do I show that?" or "How do I get from the supposition to the conclusion?" To answer these questions, I encourage students to ask themselves, "What do the supposition and the conclusion mean?" In a surprising number of cases, all we have to do is use the definitions of the various terms involved to fill in the body of the proof.

Probably the most important feature of a definition is that it has both an *if* and an *only if* direction. For instance, the term *odd* is defined as follows:

An integer x is odd if, and only if, $x = 2k + 1$ for some integer k.

Thus any time we have an odd integer, we know that it has a certain form—the *only if* direction of the definition. And any time we have an integer of a certain form, we know that it is odd—the *if* direction of the definition.

Knowing that an integer has the form $2k + 1$ for some integer k	leads to $\rightarrow$	knowing that it is odd.
Knowing that an integer is odd	leads to $\rightarrow$	knowing that it has the form $2k + 1$ for some integer k.

In the proof that the square of any odd integer is odd, the conclusion is that a certain integer, x^2, is odd. From the *if* direction of the definition, students see that if they can show that

$$x^2 = 2 \cdot (\text{some integer}) + 1,$$

then they will be able to complete the proof. But by the hypothesis that x is odd and the *only if* direction of the definition, they can deduce that

$$x = 2k + 1$$

for some integer k. So, by substitution, the problem reduces to showing that

$$(2k + 1)^2 = 2 \cdot (\text{some integer}) + 1,$$

which leads to the algebraic steps that are performed in the proof and answers the last remaining question, question 4.

WHAT ABOUT DISPROOF?

The main purpose of teaching proof is to give students a tool for analyzing the truth or falsity of mathematical statements. It follows that it is just as important to teach students how to disprove statements as how to prove them. The basic method used to disprove most mathematical statements is the same as that used to show the falsity of the statement "For all students x, if x is in this class, then x is over twenty," namely, find one student, x, who is in the class but who is not over twenty.

In general, a statement of the form "For all x in D, if $H(x)$, then $C(x)$" is disproved by finding an element x in D for which $H(x)$ is true and $C(x)$ is false. Such an x is called a *counterexample*. For instance, consider the statement "For all real numbers x, if $x^2 > 4$, then $x > 2$." The number -3 is a counterexample that disproves this statement because $(-3)^2 = 9$ and $9 > 4$ but $-3 \not> 2$.

WHAT ABOUT PROOF BY CONTRADICTION?

Students find proof by contradiction considerably harder to master than direct proof, so it is especially important to link the basic idea of the method to students' previous experience. When I introduce the concept, for instance, I might begin by asking the class, "How do you know that today is not Thanksgiving?" Various responses usually come back: "We would be home eating turkey," "We would not have class," or "It would be Thursday." "Ah!" I reply, "so if it were Thanksgiving, various other things would have to be true, and since they are not, we know it is not Thanksgiving." Yes, that is right. That is what *proof by contradiction* is all about.

> To prove by contradiction that a statement is true,
>
> *suppose* that it is false
>
> and then
>
> *show* that this supposition leads logically to a contradiction.

Note that both fundamental methods of proof—direct and by contradiction—start by supposing one thing and conclude by showing something else. For both methods a student need ask only "What am I supposing?" and "What must I show?" realizing, however, that the answers will be very different in each case.

One of the most serious difficulties that students have in actually constructing proofs by contradiction on their own is in supposing the wrong thing. For instance, consider the following common "proof" that the product of any nonzero rational number and any irrational number is irrational:

PROOF. Suppose not. Suppose that the product of any nonzero rational number and any irrational number is rational. But 1 is a nonzero rational number and $\sqrt{2}$ is an irrational number, and their product is $1 \cdot \sqrt{2} = \sqrt{2}$, which is irrational. This example contradicts the supposition that the product is rational. Hence the product of any nonzero rational number and any irrational number is irrational.

The problem here is that in starting the proof by writing the negation of—

> The product of any nonzero rational number and any irrational number is irrational,

the student wrote—

> The product of any nonzero rational number and any irrational number is rational

instead of—

> There exist a nonzero rational number and an irrational number whose product is rational.

A teacher who knows how prone students are to make this mistake can be sure to give lots of practice in taking negations of the various forms of statements that arise in mathematics: those involving *and, or, if-then, for all,* and *there exists*. Even when some other instinct overcomes them and they write the negation incorrectly, at least the teacher can point out the error and they will almost always understand. When students have not had practice negating statements, it is very difficult to convince them that the foregoing "proof" is invalid.

A UNIFIED FRAMEWORK

Presenting proof and disproof as outlined builds a unified framework for solving a large class of mathematical problems. Faced with a statement of the form "For all x in D, *if* $H(x)$, then $C(x)$," whose truth or falsity is unknown, we reason as follows: Suppose that x *is* an element of D for which the hypothesis $H(x)$ is true. Must $C(x)$ also be true? If we show that the answer is "yes," then we have a direct proof. If we show that the answer is "not necessarily," then we have a disproof by counterexample. If we show that it is impossible for $C(x)$ to be false, then we have a proof by contradiction.

And that is it: direct proof, disproof by counterexample, and proof by contradiction are three aspects of the same whole. We arrive at one or another by a thoughtful examination of the statement in question, knowing what it means for a statement of that form to be true or false.

Learning to disprove statements is just as important as learning to prove statements

EXPANDING THE USE OF PROOF IN HIGH SCHOOL

High school students can develop their abilities to prove and disprove mathematical statements in many settings besides a geometry course and the formal presentation of mathematical induction. Ideally, proof and disproof should be a theme running through the entire secondary mathematics curriculum, with a foreshadowing of the ideas in the middle grades. Of course, teachers should start gradually, perhaps first giving students just questions that can be answered by finding a counterexample; next adding problems that can be solved by citing a known property, such as the distributive law; then asking questions that require a direct proof; and finally introducing proof by contradiction.

To make these ideas accessible to students of a wide range of ability levels in my own courses, I try to offer a varied mix of activities in virtually every class period: exercises in logic to lay the basis for working in an abstract setting, concrete exploratory exercises to give practice working with the definitions involved in the various statements we discuss, and applications exercises to elucidate the value of what we are doing. When introducing a new form of proof, I also present fill-in-the-blank exercises to help students get used to the flow of the proof before I ask them to construct proofs entirely on their own.

Examples of problems that could be used at the high school level follow. Similar problems can be found in many college discrete mathematics textbooks, transition-to-higher-mathematics textbooks, and the high school textbook *Precalculus and Discrete Mathematics* developed as part of the University of Chicago School Mathematics Project (Peressini et al. 1992).

The "generic particular" mode of thought is at the heart of all abstract mathematics

Algebraic formulas. Is the formula $(x + y)^2 = x^2 + y^2$ true for *all* real numbers x and y? For *some* real numbers x and y? For *no* real numbers x and y? Explain.

Properties of even and odd integers. True or false? The sum of any two even integers is even. Justify your answer.

Divisibility properties of integers. True or false? For any integers a, b, and c, if bc is divisible by a, then b is divisible by a or c is divisible by a.

Properties of rational numbers. True or false? The product of any nonzero rational number and any irrational number is irrational.

One-to-one and onto functions. Let $f(x) = x/(x^2 + 1)$. Is f one-to-one? Is f onto the set of all real numbers? Explain.

Properties of logarithms. If b and y are positive real numbers with $\log_b y = 3$, what is $\log_b (1/y)$? Why?

Increasing and decreasing functions. Let $g(x) = (x - 1)/x$ for all real numbers $x \neq 0$. Is g increasing for all real numbers $x > 0$? Justify your answer.

Properties of trigonometric functions. True or false? For all real numbers x, $\sin^2(x) = 1 - \cos(2x)$. Explain.

High school teachers can make other choices that can affect the extent to which their students develop deductive-reasoning abilities. For instance, the equation of a straight line is commonly introduced as a formula for students to memorize. This approach may well be appropriate for students' initial introduction to the topic. But later on, at least, they could be exposed to the classic method, which is actually based on the idea of generalizing from the generic particular. Here is how this method works: What makes a straight line straight is that no matter what two distinct points are chosen on it, the slope computed by using them is the same as that computed by using any other two distinct points chosen on it. This result, of course, follows from properties of similar triangles.

Suppose that we want to find the equation of the line with slope –2 that passes through the point (1, 3). See **figure 1.** We imagine (x, y) to be any other (particular but arbitrarily chosen!) point on the line then compute the slope by using (x, y) and (1, 3). Since the result must equal –2, we have the equation

$$\frac{y-3}{x-1} = -2.$$

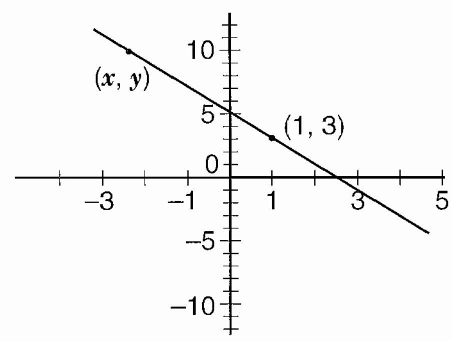

Fig. 1
Finding the equation of the line with slope –2 that passes through (1, 3)

When both sides are multiplied by $x - 1$, the result is

$$y - 3 = -2(x - 1),$$

which is true not only when $(x, y) \neq (1, 3)$ but also when $(x, y) = (1, 3)$, which we see by plugging $x = 1$ and $y = 3$ into the equation. Hence *every* point on the line satisfies this equation. Conversely, every point satisfying the equation is on the line, so this equation *is* that of the given line. From a practical point of view, this method for finding the equation of a line is as easy to use as a formula. But the thought processes used in applying it are very different. Each time a student uses the classic method, he or she gains practice in reasoning deductively and employing the "generic particular" mode of thought that is at the heart of all abstract mathematics.

CONCLUSION

Those of us to whom mathematical thinking comes easily find that it is hard to realize all the many small thought processes that our students must master to engage effectively in mathematical abstraction. For most of our students, becoming comfortable with the logic of mathematical thought does not happen overnight. But with steady support from us, they can make significant progress. The Russian psychologist L. S. Vygotsky pointed out that what students are able to do when they work entirely on their own is very different from what they can accomplish with the guidance of a teacher; the work they do with our help today enables them to achieve success tomorrow on their own (Vygotsky 1935).

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