THE LOGIC OF TEACHING CALCULUS

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The Problems

Eight years ago my department instituted a course in mathematical reasoning to serve as a transition between calculus and higher-level math classes. We had found that students were entering our higher-level classes woefully unable to construct the most simple proofs or to figure out answers to easy abstract questions. The idea of the course was to give students a better chance for success in more advanced classes (1) by teaching the basic techniques of mathematical proof in such a way that students would learn to use them themselves, and (2) by spending an adequate amount of time on the rudiments of set theory, equivalence relations, and function properties rather than hurrying through these topics quickly as often happens at the beginning of advanced courses.

Over the next several years I had primary responsibility for developing the course. During this period I came to realize that many of my students' difficulties were much more profound than I had anticipated. Quite simply, my students and I spoke different languages. I would say "Of course, this follows from that" or "As you can see this means the same as that" and my students would look at me blankly.

Very few of my students had an intuitive feel for the equivalence between a statement and its contrapositive or realized

that a statement can be true and its converse false. Most
students did not understand what it means for an "if-then"
statement to be false, and many also were inconsistent about
taking negations of "and" and "or" statements. Large numbers used
the words "neither-nor" incorrectly, and hardly any interpreted
the phrases "only if" or " necessary and sufficient" according to
their definitions in logic. All aspects of the use of quantifiers
were poorly understood, especially the negation of quantified
statements and the intepretation of multiply-quantified
statements. Students neither were able to apply universal
statements in abstract settings to draw conclusions about
particular elements nor did they know what processes must be
followed to establish the truth of universally (or even
existentially) quantified statements. Specifically, the technique
of showing that something is true in general by showing that it is
true in a particular but arbitrarily chosen instance did not come
naturally to most of my students. Nor did many students
understand that to show the existence of an object with a certain
property, one should try to *find* the object.

The conclusions I came to through observing my students are
in substantial agreement with the results of systematic studies
made by modern cognitive psychologists. As the British
psychologist P. N. Johnson-Laird put it in 1975: "It has become a
truism that whatever formal logic may be, it is not a model of how
people make inferences." [5] A common estimate is that under 5%
of people use "correct" logic spontaneously. Even Piaget in his
later years came to modify his view that the development of formal

modes of thought was a natural occurrence at a certain stage of
adolescence and acknowledged that his original work had been based
on a "somewhat priviledged population." [8]

In a course such as mine, the consequences of such poor
intuition for logic and language were devastating. For example,
at one point in the course students were asked to prove that the
sum of two rational numbers is rational. Consider what thought
processes are involved in creating such a proof. Here is a
partial list.

(1) One must understand, either consciously or subconsciously,
that the statement is universal, that it says something about
all pairs of rational numbers.

(2) One must realize that to prove this universal statement is
true, one supposes one has two particular but arbitrarily
chosen rational numbers and shows that *their* sum is rational.
(That is, one must understand either consciously or
subconsciously the method of proof using the "generic
particular.")

(3) One must know both that if a number is rational then it can
be expressed as a quotient of integers and also that if a
number can be expressed as a quotient of integers then it is
rational. (That is, one must understand how to use both
directions of a definition: the "if" and the "only if." Also
it is helpful to associate a vision of a blurry fraction with
the term rational.)

(4) One must understand the rule for adding fractions as an
abstract universal truth that can be applied in an general
algebraic setting.

(5) Since virtually every step in the proof is a conclusion of a syllogism, one must understand how conclusions follow in syllogistic reasoning by applying universally applicable facts to particular instances.

At another point in the course, students were asked to prove by contradiction that the negative of an irrational number is irrational. To succeed at this task, one must realize that if the given statement is false then there is an irrational number whose negative is rational. (That is, one must be aware at some level of consciousness that the negation of a universal statement is existential.) Also, of course, one needs a sense for the logical flow of proof by contradiction.

At still another point in the course, students were asked to prove that the composition of one-to-one functions is one-to-one. To construct a proof of this statement, a really sophisticated ability to instantiate is necessary. One must understand that when a function f is one-to-one, the statement "if $f(x_1)=f(x_2)$ then $x_1=x_2$" holds for *all* x_1 and x_2 *even when* x_1 happens to be called $g(x_1)$ and x_2 happens to be called $g(x_2)$.

As noted, the above understandings need not be at a conscious level. Lots of working mathematicians have never studied formal logic and get along just fine. Unfortunately, that is part of the problem. On the one hand we have the professor for whom formal reasoning is second nature and who is usually not even consciously aware of the formal logical components of mathematically correct arguments. And on the other hand we have a mass of students for most of whom hardly any of the logical component elements of

arguments are understood on an intuitive level. The lack of insight of the professor to students' problems with logic and language are manifested in many ways. For example, it is common nowadays to omit the words "only if" in formal definitions. Supposedly this is in the interest of "simplicity." In fact, in my experience, if one hopes to impart to students a useful working knowledge of a definition, it is not only necessary to state the definition in "if and only if" form but also to state the "if" and the "only if" directions as separate sentences and to emphasize the universal character of each direction. For example, in giving the definition of rational number one needs to explain both that whenever a quantity in a discussion is known to be rational then it must be a quotient of two integers and also that whenever a number is known to be a quotient of two integers then one can infer that it is rational. Nor is it sufficient to state that an irrational number is one that is not rational. One must go on to explain that this means the number cannot be expressed as a quotient of any two integers.

Similarly, it is common in mathematical writing to leave out or to veil the presence of universal and existential quantifiers. As Alan Bundy states when introducing the concept of quantifier in his book *The Computer Modelling of Mathematical Reasoning* [2]: "Variables in mathematical expressions often have ambiguous status, whose resolution depends on the context." He then compares the x's in the two sentences

$$(x-y)(x+y) = x^2 - y^2$$

and

$$\text{Solve } x^2 + 2x + 1 = 0 \text{ for } x$$

explaining that in the first case the universal usage is
ordinarily intended while in the second case the existential usage
is meant.   He next gives an example of a single sentence in which
the two usages are combined:

$$Solve \ ax^2+bx+c = 0 \ for \ x.$$

Now to a mathematician this problem is perfectly clear.   But to a
high school algebra student the status of the variables may seem
mysterious indeed.

Implications for the Teaching of Calculus

        The implications of these observations for the teaching of
calculus are profound.   Calculus has so many definitions, so many
theorems, so many applications, so much notation, so much "abuse
of language" (as the French call it), so much logical complexity,
so much abstraction.   Consider a student who does not even know
that to prove a sum of two rational numbers is rational one starts
with two arbitrarily chosen rational numbers.   How can such a
student begin to understand even the most "intuitive" explanation
that the limit of the sum of any two functions (which have limits)
is the sum of their limits?   Not to mention the fundamental
theorem of calculus!

        It seems that most mathematics professors at most colleges
and universities are aware of an intellectual gulf between
themselves and their students.   Last year at the Joint Mathematics
Meetings the Association for Women in Mathematics sponsored an
panel which featured five mathematicians who had left academia for
employment in business or industry.   One panelist after another
spoke of being disillusioned with the quality of student they had
had to teach during their periods as academics.   Maria Klawe

(Discrete Mathematics Manager at the IBM San Jose Research Laboratory) was especially eloquent as she spoke of her despair and frustration at the thought that all the years she had spent in graduate school developing her capacity for abstract thought would be wasted in an effort to teach students who would not and could never learn college mathematics. While it was not too surprising that the panelists spoke so disparagingly (they had after all chosen to leave classroom teaching), what was surprising was the loudly sympathetic reaction of the large audience.

Professors react to the gulf between themselves and their students in different ways. One reaction is to ignore it, to state the definitions and prove the theorems as if the students would understand them and were mature enough to be able to derive simple consequences (such as problem solutions) on their own. This approach is often associated with the calculus instruction of 20 to 30 years ago and is fondly remembered by many mathematicians. (It is bitterly remembered by many physicists and engineers.) Another response, widely adopted today, is to expose the basic concepts of calculus at a moderately high level, emphasizing intuition, but focus primarily on skills, and only test students on their ability to perform certain mechanical computations in response to certain verbal cues. In this approach the vast majority of students indulge their professors by listening quietly to their often inspired and beautifully intuitive explanations and the professors repay their students' courtesy by making their explanations brief, spending lots of time demonstrating procedures to solve rote problems, and never asking students to do anything on an exam that requires genuine knowledge

of concepts.

        Of course, there are good reasons for giving attention to the
mechanical aspects of calculus. The best is based on the sound
pedagogical promise that understanding in mathematics comes in
pieces. Often, learning to use certain techniques mechanically
helps one progress to conceptual understanding. But this approach
becomes perverted if in practice conceptual understanding is
indefinitely postponed. Another benefit of emphasizing mechanics
is that such activities as practice in formal differentiation and
integration improve students' pattern recognition skills. A
possible third argument in favor of an emphasis on calculus
mechanics is to prepare students for courses in physics and
economics and engineering. But I won't make this argument. For
too many years I have heard complaints from my colleagues in other
departments about the mathematical knowledge of the students we
send them. Invariably, the "simple" examples they give as
evidence that our students can't perform involve being able to
think, not just compute on cue.

        The fact is that the state of most students' conceptual
knowledge of mathematics after they have taken calculus is
abysmal. The most dramatic formal studies on this subject have
been done by John Clement, Jack Lochhead, and others in the
Cognitive Development Project at the University of Massachusetts
at Amherst. They found that a large majority of calculus and
post-calculus students tested at universities throughout the
country could not set up or even correctly interpret simple
proportionality equations. In summarizing the results of their
many experiments, Lochhead wrote: "many college students are not

facile at reading or writing simple algebraic equations" and at a deeper level "students seem to lack any well defined notion of variable or of function." [6] Currently Hadas Rin is studying student difficulties with calculus by examining their spontaneous written questions. Among her findings are the misuse of course vocabulary by students (for example, "How do you find the tangent to the slope?" or "Any number has no derivative"), the inability to instantiate "known" theorems in new situations (for example, asking for "the rule" to differentiate a function of the form $\frac{f \cdot g}{h}$), and lack of understanding of definitions, not just of sophisticated concepts such as limit but also of more fundamental ones such as secant and tangent. [9]

To me it seems incontrovertible that the primary aim of calculus instruction should be the development of conceptual as opposed to purely mechanical understanding. In this computer age, software packages are now or soon will be available to perform any standard calculus computation including taking symbolic derivatives and integrals and even testing series for convergence. People using these packages need some computational facility themselves (just as experience doing arithmetic by hand is needed for a person to make best use of a calculator). But the main requirement to use calculus packages effectively is firm conceptual understanding of the subject matter. With computers to take care of mechanical details, the premium is on the abilities to abstract, to infer, and to translate back and forth between formal mathematics and real world problems. Yet these are abilities of the highest order, normally associated with a small number of students of exceptional talent.

Suggestions

Never in history have mathematicians been called upon to
teach so much mathematics to so many students. Under these
circumstances, it should not be surprising that new pedagogical
methods may be necessary. The shortcuts and gaps that can be
followed and filled in by students of unusual ability may not be
negotiable for those less fortunate. To a much greater extent
than is currently the case, there is a need to respond to
students' lack of sophistication, not by giving up but by helping
them.

One possibility is to modify precalculus courses to make them
include additional work to increase students' logical maturity. I
see this as a potential benefit of the movement to introduce
discrete mathematics early in students' undergraduate careers.
Within the context of a course in calculus, I would suggest the
following measures. Some (perhaps all) may be controversial. I
have found all of them useful.

(1) State logically complex sentences (such as the definition
of limit) and pose problems in a variety of equivalent ways. Left
to themselves, students usually do not turn concepts over in their
minds to view them from many angles. For instance, one could ask
students to

"Describe the values of the expression $\dfrac{(x+1)}{(x-1)^2}$ when

x takes values very close to 1"

as well as to

"Find $\displaystyle\lim_{x\to 1}\dfrac{(x+1)}{(x-1)^2}$."

(2)   When lecturing, write more or less in complete sentences. When the words "if-then" or "for all x in the interval [a,b]" are not written out, they will not appear in students' notes, nor will they be supplied mentally.

(3) Make an effort to clarify statements whose quantification is implicit. For example, the implicit quantification of the phrase "solve the equation" goes hand-in-hand with a mechanical approach based on formal symbol manipulation rather than a conceptual approach based on studying numbers and their properties. Leon Henkin suggests that teachers of beginning algebra students avoid using phrases like "Solve the equation $2x+3 = 0$" and instead say "Find a number x such that $2x+3 = 0$." And instead of "Solve $ax+b = 0$" he suggests "If a and b are numbers and $a \neq 0$, find a number x such that $ax+b = 0$." [4] Or, instead of asking students to solve equations, one could ask them questions like "Are there any real numbers such that $x^2-3x+2 = 0$?" or "such that $x^2-x+1 = 0$?" or "such that $\sqrt{x+1} + \sqrt{2x+1} = 2$?" College calculus students would also benefit from occasionally having problems phrased in these ways.

(4)   Avoid unnecessary notation and terminology. For most students, each mathematical term and symbol is a hurdle to be crossed. Let's not put any more in their way than we have to.

(5)   Try to avoid notational and linguistic "abuses" as much as possible. In the long run, it is worth the extra effort to say "Let f be the function defined by the rule..." rather than "Let f(x) be the function..."

(6)   Frequently clarify lines of argument by explaining the underlying logic.

(7) Make students memorize precisely-worded definitions and perhaps theorem statements also. Memorization is greatly underrated as a pedagogical tool. At the least, memorization of a definition or theorem forces students to read it carefully; at best, it encourages them to understand it (since it is easier to memorize something intelligible than gibberish). Also memorization of precise language gives students experience in using it and makes it necessary for them to pay attention to words like "if" and "then" that they might otherwise ignore.

(8) Develop or seek out problems to act as cognitive bridges to abstract understanding. I have found, for example, that students are fairly capable of understanding concepts in purely geometric terms. They do not seem to have problems learning to distinguish between the graphs of continuous and discontinuous functions or between concave up and concave down. The difficulties arise when analysis is added to the picture. One reason is that, in my experience at least, beginning calculus students do not know the abstract definition of graph of a function. They can plot and connect points for a function given by a specific formula, but they do not know that for a general function f, $f(x)$ is the height to the graph of f at x. Now since most calculus explanations are given in terms of "generic" functions f and "generic" points x, this seriously inhibits students' ability to follow text and classroom explanations.

To counteract this difficulty, I would suggest adding problems of the following type to the usual collection on graphing.

Let f and g be function defined for all real numbers.

(a) Suppose f(4)=9. What point must lie on the graph of f?

(b) Suppose the point (-1,2) lies on the graph of f. What can be inferred about f?

(c) Suppose the point (3,g(3)) lies on the graph of f. What can be inferred about the relation between f and g?

(d) Suppose the graphs of f and g have a point in common, say (x_0, y_0). What can be inferred about the relation between $f(x_0)$ and $g(x_0)$?

Later, just prior to the introduction of the analytic definition of the slope of the tangent line, one would assign exercises such as these.
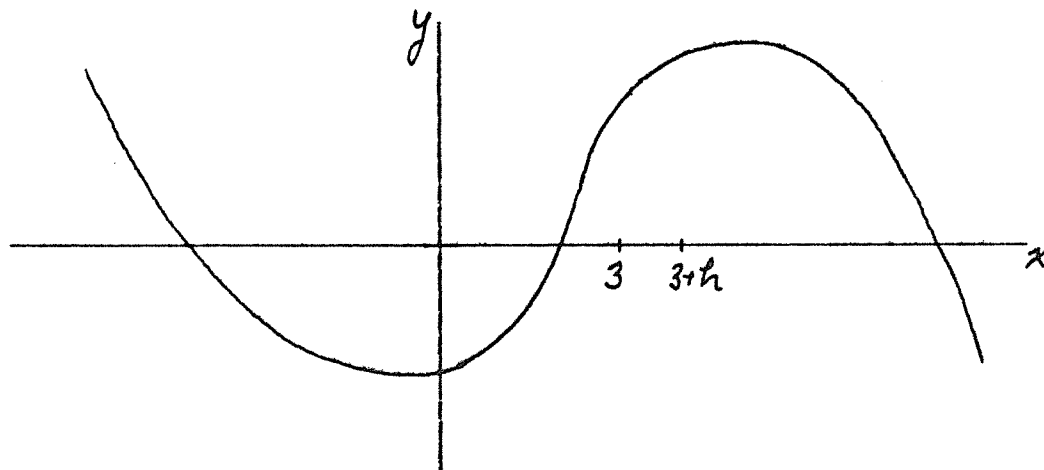
1. Let f be the function whose graph is given below.

   (a) Label the points (2,f(2)) and (4,f(4)) on the graph and draw the secant line through these two points.

   (b) Find an expression for the slope of the secant line through the points (2,f(2)) and (4,f(4)).
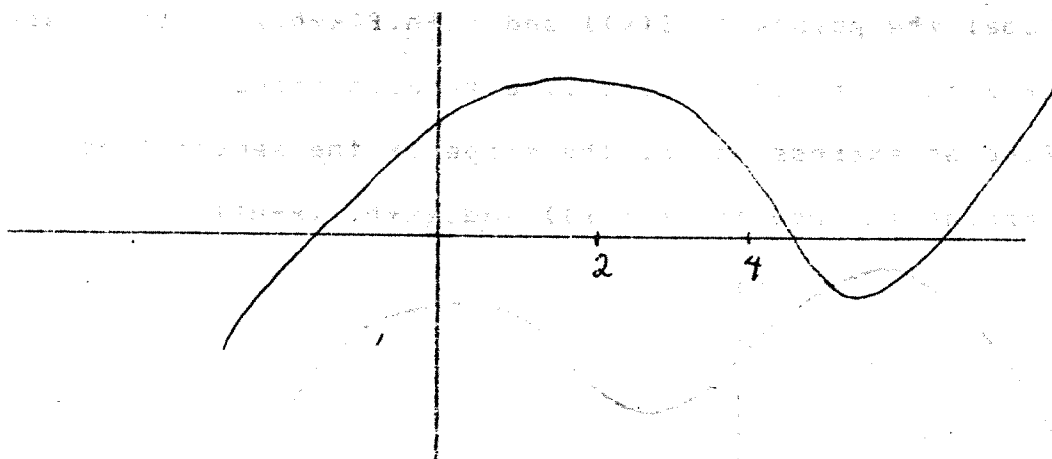
2.   Let f be the function whose graph is indicated below and

suppose h is a (small) positive number.

   (a) Label the points (3,f(3)) and (3+h,f(3+h)) and draw the

       secant line through these points.

   (b) Find an expression for the slope of the secant line

       through the points (3,f(3)) and (3+h,f(3+h)).

3.   Let f be the function whose graph is indicated below.

Suppose also that x is a number and h is a positive number and

f takes values at x and x+h.

   (a) Label the points (x,f(x)) and (x+h,f(x+h)) on the graph

       of f and draw the secant line through these points.

   (b) Find an expression for the slope of the secant line

       through the points (x,f(x)) and (x+h,f(x+h)).
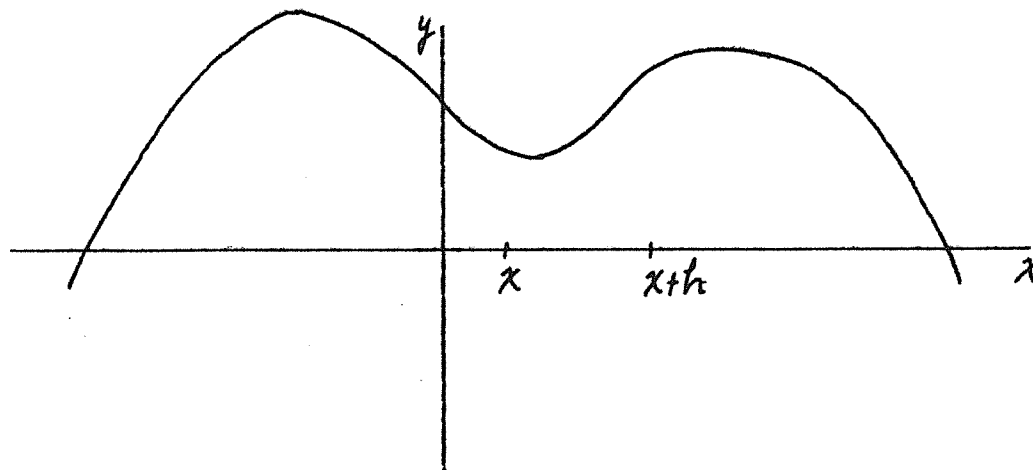
4. Let x be the number represented by the labeled point on the number line below and suppose h is a *negative* number. Indicate and label a reasonable choice of point to represent the number x+h.

$$\underset{x}{\rule{12cm}{0.4pt}}$$
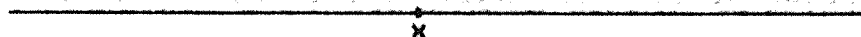
(9) Do not be satisfied with narrow understanding of abstract principles. Throughout their mathematical careers students have great difficulty learning universal facts in their generality. (Logically speaking, they have difficulty instantiating universal statements over the full extent of their domains.) Much mathematics instruction takes these problems into account. For example, precalculus texts often have exercises that are graded with problems like "Factor x^2+5x+4" in the "A" set, "Factor $3x^2-14x+8$" in the "B" set and "Factor $24x^2-31xy-15y^2$" in the "C" set. [1] In calculus, also, concepts can be understood at "A," "B," and "C" levels of generality. For example, in a study of the chain rule an "A level" problem would be

"Find $\frac{dy}{dx}$ for $y = (3x+2)^2$,"

a "B level" problem would be

"Find $\frac{dy}{dx}$ for $y = \sqrt{\sin 2x}$,"

and a "C level" problem would be

"Assume f is a differentiable function. Let $y = [f(x)]^4$ and suppose $f'(1) = 5$ and $\frac{dy}{dx}\Big|_{x=1} = -160$. Find $f(1)$. [3]

It is ironic that the same students who, as second graders, were *judged needful of spending a full year on two-place addition*

before moving to the complexity of the three-place case are as 18-year olds expected to generalize in one homework assignment from "A level" chain rule problems to "C level" ones. Perhaps problems of increasing difficulty could be assigned and discussed over a period of several days, concurrently with other topics if necessary, to allow time for the abstraction process to occur.

(10) Include questions that test conceptual understanding on homework and on exams. In [7] Jean Pedersen and Peter Ross make some excellent suggestions of such problems, which test understanding both of geometric and analytic aspects of concepts.

(11) Take responsibility for all aspects of students' mathematical development. We do not help our students when we ignore "mere" algebra mistakes just because algebra is not the subject of our course.

(12) Give students opportunities to speak and write using the course vocabulary.

(a) Insist that students give complete, coherent answers to questions on exams. No favor is done students' intellectual development by giving full credit for a few scribbles.

(b) Have students present solutions to problems at the board occasionally, and insist that they explain their work aloud to the rest of the class. You may be appalled by their misuse of language, but don't despair. Just correct the worst mistakes courteously and find something to praise. Most people do not learn to speak a foreign language by attending lectures and doing grammar exercises. They have to make fools of themselves by

talking out loud. The same goes for learning the language of
mathematics.

(c) Occasionally break classes up into small groups for
collective problem-solving experiences. Even if this is done only
once or twice a semester, it can have impact on students' ability
to put their mathematical thoughts into words. Collaboration also
encourages students to explore new ideas more boldly than they
would on their own.

(d) Every once in a while restrain the impulse to give the
answer to a question on the homework as soon as it is asked.
Instead, open the question up for class discussion. (This works
only for carefully selected problems.) In one of the liveliest
classes I have ever taught, a student asked me to answer the
following homework question.

"Determine whether the following statement is true, and
support your answer by giving a proof or a counterexample:

If a, b, and c are integers and a/b and a/c then $a/b \cdot c$."
Instead of complying, I told the class to imagine they were the
mathematical research division of a large company and had been
given the job, as a group, of figuring out the answer to this
question. Important decisions depended on the answer and the
company was counting on its being correct. In the discussion that
followed, I acted as moderator. I guided, but I did not reveal
the answer. At one point the board contained two false proofs, a
false counterexample, and a true counterexample. (I am happy to
report that, overwhelmingly, the students were convinced by the
true counterexample when it finally appeared.) During the

discussion almost all the logic we had covered to that point in the course was reviewed and its importance to the determination of the mathematical truth of the situation was apparent. There was also a lively follow-up discussion on the nature of mathematical discovery.

Conclusion

If all the suggestions made above were incorporated into the teaching of calculus, it would probably be impossible to cover as many topics as is now standard. The question is: What is the trade off? If it comes to a choice, will we settle for superficial knowledge of a lot or deeper understanding of less? Perhaps less is more.

Do I think that by following these suggestions a new breed of mathematical super-students will be created? Certainly not. But I do think there are many students "out there" of reasonable mathematical aptitude for whom mathematics is more mysterious than it needs to be. These are the students we can affect with better pedagogy. I think it is worth trying.

References

[1] Barnett, Raymond A., *Intermediate Algebra: Structure and Use*, Second Edition. New York: McGraw-Hill, 1980.

[2] Bundy, Alan, *The Computer Modelling of Mathematical Reasoning*. London: Academic Press, 1983.

[3] Fraleigh, John B., *Calculus with Analytic Geometry*, Second Edition. Reading, Massachusetts: Addison-Wesley, 1985.

[4] Henkin, Leon, "Linguistic Aspects of Mathematical Education." In *Learning and the Nature of Mathematics* edited by William E. Lamon, pp. 211-218. Chicago: Science Research Associates, 1972.