

Logic and Discrete Mathematics in the Schools

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Albert Einstein once said [3], “the whole of science is nothing more than a refinement of every day thinking.” This quotation aptly summarizes the essential interdependence between the concrete and commonsensical and the abstract and theoretical. Developing students’ abilities to shift smoothly and flexibly between these two levels, while operating effectively on each one, is arguably the central task of mathematics and science instruction.

In the language of the NCTM *Standards* [6], a primary goal of mathematics instruction should be to develop students’ “mathematical power,” which is the ability “to explore, conjecture, and reason logically, as well as . . . to use a variety of mathematical methods to solve nonroutine problems.” As Uri Treisman and Dick Stanley have put it [9], mathematics instruction should “concentrate less on the low-level use of high-level ideas and more on the high-level use of low-level ideas.”

Those involved in the mathematics education reform movement have identified various specific elements that contribute to developing students’ higher-level reasoning skills, such as experience working with open-ended and slightly ill-formed problems, opportunities for learning to perceive mathematical issues in a broad variety of different contexts, practice in recognizing the need for and in providing justification for mathematical assertions, increased use of cooperative learning, and employing calculators and computers to provide answers to routine parts of problems.

But success in these reformed mathematical environments requires that students—whether they know they are doing so or not—correctly apply the laws of classical logic in a variety of different settings. Specifically, in order to be able to reason effectively, students need to know

- that just because a statement of the form “if p then q ” is true, one cannot conclude that “if q then p ” is also true (or “if not p then not q ”, for that matter);
- that another way to phrase a statement of the form “if p then q ” is “if not q then not p ”;

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- what it means for statements of the following form to be false: “ p and q ,” “ p or q ” and “if p then q ”;
- that if a property fails to hold in just one instance, then it does not hold universally;
- that to show a property holds universally, one shows that it holds in a particular but *generic* instance;
- that certain forms of argument are inherently erroneous (invalid) whereas other forms can be trusted to produce true conclusions if given true premises.

The Problem

Unfortunately, research by cognitive psychologists strongly suggests that the vast majority of students do not develop the reasoning skills described above during their high school years. Moreover, although a small proportion of the population (approximately 4%) appears seemingly spontaneously to develop a good capability for formal reasoning, most of the population does not [1]. Thus most students, both in high school and college, need assistance in order to improve their ability to think logically.

Perhaps in an ideal world the fundamentals of logical reasoning would be adequately conveyed to students in the context of studying other topics by teachers who know how to seize the “teachable moment” and who recognize the importance of instilling general principles of reasoning in students’ minds. But the world is not ideal. For one thing, most of us are less adept at catching that elusive moment than we would wish. But, more importantly, if logical reasoning is always presented as a subtext, in an implicit rather than explicit way, how are we to convey the expertise required to teach it from one generation to the next?

One problem is that some mathematics teachers are not completely secure in their own reasoning abilities while others take correct reasoning so much for granted that they are not able to communicate effectively with students who do not think as they do. Another problem is that when instruction in logical reasoning is not made an explicit priority, it is usually subordinated to other considerations.

An ironic consequence of the attention given to mathematics instruction over the past thirty years is that, especially in algebra, clever teachers and textbook authors have devised numerous ways to help students obtain correct answers to problems by following certain mechanical procedures rather than by reasoning them through. For instance, it used to be that students were taught to solve the problem of finding all real numbers x such that $(x + 1)(x - 2) > 0$ by applying the basic principle that a product of two real numbers is positive if and only if both numbers have the same sign. Use of this approach reinforced the notion that success in mathematics results from the intelligent application of a small number of basic principles, and it taught several important methods of logical reasoning (for instance, argument by division into cases and the logic of *and* and *or*). Nowadays,

because they didn't understand the definitions of the given functions, but because they didn't understand what it means for statements of the form displayed above to be false. That is, they did not understand (even on an intuitive level) that the negation of a universal statement is existential and that the negation of "if p then q " is " p and not q ."

After several years of experimentation, we eventually settled upon the method we still use today. Cognitive psychologists have demonstrated fairly conclusively that instruction in the abstract-principles of formal logic alone does not guarantee an increase in students' reasoning abilities [7]. Our experience also showed that in order to significantly affect cognitive processes as fundamental and broadly applicable as the correct use of the rules of formal logic, a one-shot approach is not sufficient. Just as a person's personality does not change overnight, even after the revelation and acceptance by the person of some profound personal psychological truth, neither do a person's cognitive processes undergo an instantaneous transformation even though the person may have understood and accepted (at some level) the truth of certain logical principles.

In our course, therefore, and in the book that has developed out of it [4], we use several methods to tie formal principles of logic to their use in actual reasoning situations. (Substantial portions of [8] reflect a similar approach at the high school level.) First, when we introduce the principles, we include a very large number of natural-language examples. Thus, for instance, before using truth tables to derive the law asserting that the negation of a statement of the form " p and q " is "not p or not q ," we give examples of very simple *and* statements and have students think about and discuss what the negations of these statements should be. Then, after the law has been derived formally, the bulk of the exercises ask students to apply it in natural-language situations. Later on in the course, when the law is actually used as an important step in a reasoning process, we point out its occurrence to the students. And when students occasionally use the law incorrectly in their written work, mentioning their error by name helps them better understand what they did wrong.

Student difficulties dealing with negations of universal and existential statements are handled similarly. For instance, when we introduce students to proof by contradiction (a difficult topic for most of them), we might ask students to prove that the double of any irrational number is irrational. Here is a version of a common response:

Theorem: The double of any irrational number is irrational.

Proof (by contradiction): Suppose it is not. That is, suppose the double of any irrational number is rational. But we previously proved that $\sqrt{2}$ is irrational and also that $2\sqrt{2}$ is irrational. These results contradict our supposition. Hence the theorem is true.

When the class has not previously discussed how to negate quantified statements, a teacher has great difficulty helping students understand the error in

however, a popular way to teach students to solve such inequalities asks them to learn that the solution consists of certain intervals and that if substitution of a value from one of these intervals makes the inequality true, then the interval is part of the solution. For most students, this method, while effective, serves no larger educational purpose than obtaining a correct answer to a particular problem.

Similarly, in an ideal situation, discussion of the “vertical line test” should help students deepen their understanding of the relationship between the analytic and geometric versions of the definition of a function. Instead the rule is often presented in such a way that students learn to get the right answer to the question “does this graph represent a function?” without making any real progress toward understanding what a function is. In practice, use of the vertical line test enables students to avoid dealing with the linguistic complexity of the function definition and thus fails to advance their ability to understand similarly complex statements in the future. In much the same way, students who are taught to find the inverse of a function f by solving $f(x) = y$ for x and then interchanging x and y are deprived of the opportunity to deepen their understanding both of functions and of the logic of quantified statements. Students not taught this short-cut are forced of necessity to use the definition of inverse function, learning to ask and answer the question, “given any y in the co-domain of f can I find an x in the domain so that $f(x) = y$?”

The Development of our Course

In 1978 at DePaul University we began developing a course to help students make the transition from traditional computationally-oriented mathematics to more abstract mathematical thinking. At the outset we thought that if we just gave students an opportunity to learn subject matter — such as set theory, relations, and function properties — that forms the basis of upper-level work in mathematics and computer science, they would be successful. What we discovered was that students had much more difficulty learning the material than we anticipated, and that to a great extent this difficulty resulted from a general lack of reasoning skills.

For instance, many applications involving one-to-one functions use one form of the definition:

for all x_1 and x_2 in the domain of f , if $f(x_1) = f(x_2)$ then $x_1 = x_2$,

whereas other applications use the alternate form

for all x_1 and x_2 in the domain of f , if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

When we first started teaching the course, we merely commented on the equivalence of the definitions in passing, using whichever was most convenient in any particular situation. But we soon realized that what was obvious to us (the logical equivalence of the definitions) was a major stumbling block for many of our students. Similarly, a large number of students had difficulty determining whether or not particular functions were one-to-one, not

the above “proof.” In a class where there has been prior experience working with such negations, the error is less common. And when it does occur, the teacher can recall that the class previously agreed that the existence of just one college student aged 30 or over was exactly what was needed to falsify the general statement “All college students are under 30.” Then the teacher can draw the parallel, pointing out how, in the same way, the existence of just one irrational number whose double is rational is exactly what one needs to negate the statement “the double of any irrational number is irrational.” So that is the supposition from which a contradiction must be deduced in order to prove the theorem by contradiction.

Contrast this approach with an approach in which general logical principles have never been explicitly discussed. In such a case, the teacher is in the awkward position of having to point out an error and at the same time convince the student that it really *is* an error. We find that a student who has already thought about the particular logical principle in question and at least partially accepted its validity is much more ready to integrate an appreciation for the new instance of it than a student who has never thought about the issue before. Thus by identifying a few logical principles and giving them names early in the course, we create a basis for developing a fuller understanding of them and a means by which to communicate with students about them throughout the remainder of the course.

This approach is similar to that used in both English and foreign language courses. English teachers agree that the most important part of teaching writing is having students spend time doing it. But interspersed with actual writing practice is a certain amount of explicit instruction in the rules of grammar and organization, and an important component of writing exercises is the process of correction and revision. Similarly for foreign language instruction. Before the age of about eleven, children can learn language purely by osmosis. But after the age of eleven people seem to benefit from some formal instruction in the rules of a new language as well as from immersion in it.

Outcomes

On the whole, we and our students have been very pleased with the results of our approach. We do expect at times to have to listen to and read student explanations that are quite garbled (in the early stages of discussing set theory proofs, for instance). Deeply ingrained mental habits take time to change. But what we do see is significant growth in most students as the course progresses.

For instance, we wait to discuss equivalence relations until late in the second quarter, having interspersed the more theoretical course topics with more straightforward topics and applications earlier on. The advantage is that by the time we reach this topic, the large majority of students really understand what it means for a binary relation to be or not to be reflexive, symmetric, and transitive (which requires a well-developed sense of the logic

of quantified statements, *if-then*, *and*, and *or*). The observation that a certain relation is, say, transitive “by default” is typically made with relish by several students simultaneously. And when we discuss the proof that an equivalence relation defined on a set partitions the set into a union of disjoint subsets, virtually the whole class participates in its development.

Similarly by the time we discuss the fact that any tree with n vertices has $n - 1$ edges (see Figure 1), we find that the majority of our students have sufficient familiarity with the logic of *if-then* and quantified statements to comprehend the subtlety of the proof by mathematical induction. The difficulty in the proof comes in understanding why the proof of the inductive step proceeds as it does. In our course, the structure of the proof is seen as a natural consequence of the general logical principle that to prove a statement of the form

for all elements in a set, if (*hypothesis*) then (*conclusion*),

one assumes that one has a (particular but arbitrarily chosen) element of the set which makes the hypothesis true, and one shows that this element makes the conclusion true also. That is why in the proof of the inductive step one assumes that k is any positive integer for which property $P(k)$ holds (that is, one assumes that any tree with k vertices has $k - 1$ edges), and then one shows that $P(k + 1)$ must also hold (that is, one shows that any tree with $k + 1$ vertices has k edges). Moreover, to show that any tree with $k + 1$ vertices has k edges, application of the same logical principle leads one first to suppose that T is any (particular but arbitrarily chosen) tree with $k + 1$ vertices and then to show that (this particular) T has k edges.

Even after so many years of intimate connection with this course, I am still amazed that students who are clearly bright by many measures and have done extremely well in preceding parts of the course nonetheless need to take their time and feel their way with each new topic. Given encouragement, however, and the opportunity to explore, discuss, and make mistakes, such students not only succeed but they also thoroughly enjoy their success. The point is that the ability to reason with mathematics, to deduce, to justify, and to switch back and forth between abstract definitions and theorems and concrete and applied situations, is not something that students either do or do not possess. Nor is it necessarily or primarily innate. Rather it is a conglomerate of knowledge, attitudes, and tendencies whose cultivation is the greatest challenge that mathematics educators can address.

Connection with Discrete Mathematics

The primary reason for the current interest in discrete mathematics is that it provides the theoretical foundation for the technology of the information age. The ability to reason logically in abstract settings is essential for success in computer science courses at all levels of the undergraduate curriculum. Moreover, knowledge of particular topics in formal logic is indispensable for understanding the design of digital circuits and automata,

Lemma: Any tree with more than one vertex has a vertex of degree 1.

Proof: Let T be any tree with more than one vertex. Pick a vertex v at random and search outward from v on a path along edges from one vertex to another looking for a vertex of degree 1. As each new vertex is reached, check whether it has degree 1. If so, a vertex of degree 1 has been found. If not, it is possible to exit from the new vertex along a different edge from that used to reach the vertex. Because T is a tree, it is circuit-free, and so the path never returns to a previously used vertex. Since the number of vertices of T is finite, the process of building a path must eventually terminate. When that happens, the final vertex of the path must have degree 1.

Theorem: For any positive integer n , any tree with n vertices has $n - 1$ edges.

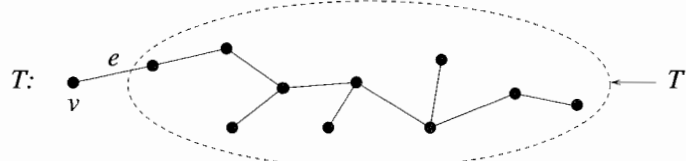
Proof: Let $P(n)$ be the property

any tree with n vertices has $n - 1$ edges

We use mathematical induction to show that this property holds for all integers $n \geq 1$.

Basis Step: Let T be any tree with one vertex. Then T has zero edges (because it contains no loops). Since $0 = 1 - 1$, the property holds for $n = 1$.

Inductive Step: We must show that for any positive integer k , if the property holds for k then it holds for $k + 1$. Let k be a positive integer and suppose the inductive hypothesis: that any tree with k vertices has $k - 1$ edges. We must show that any tree with $k + 1$ vertices has k edges. Let T be any tree with $k + 1$ vertices. Since k is a positive integer, $k + 1 \geq 2$, and so T has more than one vertex. Hence by the lemma, T has a vertex v of degree 1. Also since T has more than one vertex, there is at least one other vertex in T besides v . Thus there is an edge e connecting v to the rest of T . Let T' be the subgraph of T consisting of all the vertices of T except v and all the edges of T except e .



Then T' has k vertices, and T' is circuit-free (since T is circuit-free and removing an edge and a vertex cannot create a circuit) and T' is connected (since T is connected and removing a vertex of degree 1 and its adjacent edge from a graph does not disconnect the graph). Hence T' is a tree with k vertices, and so T' has $k - 1$ edges by inductive hypothesis. But then, since T has one more edge than T' , T has k edges.

FIGURE 1.

relational database theory, programming languages, and knowledge-based systems. Because of its importance as a topic in computer science as well as its central role in the kind of critical thinking in which computer scientists must routinely engage, logic is now a standard topic of introductory discrete mathematics courses at the college level.

Implications for the K-12 Curriculum

The majority of the reasoning skills emphasized in courses such as ours should not have to be taught for the first time at the college level. By the time students reach us, we have to expend as much effort helping them unlearn the incorrect modes of thought to which they have become accustomed as we do teaching them the correct thought processes on which mathematics is based. To achieve the lofty goals of the NCTM *Standards*, instruction in a few basic logical principles should be woven throughout the K-12 curriculum. Kindergarten is not too early for teachers to begin exploring the precise use of language with children. Indeed, even very young children can become sensitive to and enjoy making subtle linguistic distinctions. For example, starting in the primary grades, the Russian mathematics curriculum translated as part of the University of Chicago School Mathematics Project includes exercises specifically designed to develop children's logical sense [5]. In France, excellent materials have been developed for grades 6-10 for helping students make a transition to abstract mathematical thinking. (See, for instance, [2].)

In grades K-12 in the United States, however, explicit attention to the development of logical reasoning skills has been minimal or nonexistent. Our experience as described above has shown that logic can be taught explicitly and successfully within discrete mathematics. Including logic as an official topic of discrete mathematics throughout the K-12 years will not only provide a basis for more advanced study at the college level, but will help insure that the principles of formal reasoning are no longer overlooked. While there is a danger that logic will be taught in isolation, this can be avoided by well-constructed curricular materials.

The historical rationale for requiring the study of mathematics was that it sharpened the mind. Over the years this rationale has been deemphasized and greater attention has been given to the goal of acquiring specific computational skills and techniques thought to be needed in future courses or in the "real world." But the computer technology of today renders many of these computational skills less important. Using calculators and computers effectively requires general mental powers, flexibility of mind, and an understanding of concepts. Our primary goal as teachers should be to develop these abilities in our students.

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