PROOF ISSUES WITH EXISTENTIAL QUANTIFICATION

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This article focuses on issues connected with the use of existential quantification in mathematics proofs. Examples of common incorrect proofs from tertiary-level students are given, and issues raised by the proofs are analyzed: (1) the use of bound variables as if they continue to exist beyond the statements in which they are quantified, (2) the implicit use of existential instantiation, (3) the "dependence rule" for existential instantiation, and (4) universal instantiation and its use with existential instantiation. Suggestions for responding to student errors are offered.

INTRODUCTION

Many university students start a "proof" that the sum of any even integer and any odd integer is odd as follows:

> **Example 1.** *Proof:* Suppose m is any even integer and n is any odd integer. By definition of even, $m = 2k$ for some integer k. Also, by definition of odd, $n = 2k + 1$ for some integer k, and so $m + n = 2k + (2k + 1) = 4k + 1…$

To help students avoid this mistake, a reasonably effective countermeasure is to lead them to see that the resulting "proof" would not apply to *any* even integer and *any* odd integer but only to an (arbitrary) even integer and the next consecutive odd integer.

From a technical point of view, the problem with the argument in Example 1 is that it violates the logical principle known as existential instantiation:

> **Existential Instantiation**: If we know that an object exists, then we may give it a name, as long as we are not currently using the name for another object in our discussion.

In Example 1, of course, once the letter k has been used to denote the integer which, when doubled, equals m, existential instantiation prohibits giving the letter k a different meaning in the representation for n.

BOUND VARIABLES THAT EXCEED THEIR BOUNDS

The description of the mistake in Example 1 does not, however, address the issue of why students make it in the first place. One reason is that having learned the definitions of even and odd as

> An integer n is even if and only if there exists an integer k so that $n = 2k$

> An integer n is odd if and only if there exists an integer k so that $n = 2k + 1$,

students come to believe that the k's have an independent existence beyond the defining sentences. In fact, the k's are what logicians call "bound" by the quantifier "there exists." The scope of the quantifiers, and hence the binding of

the k's, extends only to the end of the sentences. In other words, the k's used in the defining sentences do not have a meaning related to even or odd integers once the definitions are finished.

Both the n and the k are, in fact, merely placeholders that make it convenient to refer, for instance, to an even integer and the integer which, when doubled, equals the even integer. Thus another countermeasure to help students avoid the mistake in Example 1 is to write the definitions of even and odd in a variety of ways – both with other letters in addition to, say, n and k and without any letters at all. For example, one can phrase the definition of even by saying that an integer is even if and only if it equals twice some integer. In general, exercises that ask students to phrase mathematical statements both formally (using variables and quantifiers) and informally (avoiding the use of variables as much as possible) are very helpful in deepening their understanding of the use of variables in mathematical discourse.

Sometimes we have to acknowledge that mathematicians' own use of notation or terminology has unforeseen consequences for student understanding. Indeed, another reason students may make the mistake in Example 1 is that they have seen instances in ordinary mathematical writing which appear to suggest that variable names can, in fact, maintain their meaning beyond the statements in which they are bound. For instance, when the statement of a theorem is universal, the variables in the hypothesis are bound. From a technical point of view, therefore, the proof should start by introducing the hypothesized objects as generic elements, say by writing something like "Suppose that…" or "Assume that…" or "Given …."[1] Yet it is common practice to view this step as unnecessary repetition. The following is a simple example:

> **Example 2.** *Theorem:* If n is any odd integer, then n^2 is odd.
> *Proof:* Since n is odd, there is an integer k such that $n = 2k + 1$. Therefore, $n^2 = (2k + 1)^2 \ldots$

IMPLICIT USE OF EXISTENTIAL INSTANTIATION

The second sentence in Example 2 illustrates another subtle but important phenomenon related to existential instantiation. Consider the statement "There is an integer k such that $n = 2k + 1$." Even though the k in this statement is a bound variable, the very fact of using the specific letter k encourages a reader to imagine a particular integer, called k, that satisfies the equation. In other words, because the statement names the integer k, it is common to proceed as if existential instantiation had already been used to bring a specific integer into the discussion and call it k. This is what occurs in the second sentence of Example 2 where k is treated as an instantiated object.

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[1] This follows from the logical principle known as Universal Generalization: If we can prove that a property is true for a generic element of a set (i.e., a particular, but arbitrarily chosen, element of the set), then we can conclude that the property is true for every element of the set. (As a proof technique, this property is also called Generalizing from the Generic Particular.)

THE DEPENDENCE RULE

Following common usage, let us call an AE statement one in which a universal quantifier precedes an existential quantifier. As the analysis of Example 2 suggests, the use of the same symbol in both the existential part of an AE statement and in a subsequent existential instantiation of the existence part of the statement is extremely common in mathematical writing. In many cases it causes no problems, but in some it leads to error.

Arsac and Durand-Guerrier (2005) introduce the term ***dependence rule***, to describe the fact that in the AE statement "For all x in set D, there exists a y in set E…" the value of y depends on the value of x. In other words, if x is changed, y must ordinarily be changed also. Their main point is that as soon as an argument contains two AE statements with elements of the same sets, the dependence rule becomes critical. Thus a refinement of the explanation for the mistake in Example 1 is that since the value of the k in the second sentence depends on m, it is highly unlikely to be the same as the value of k in the third sentence, which depends on n. This affords an additional way to help students understand the mistake they make when they start a proof as in Example 1.

Making students aware of the dependence rule can also provide a means for responding when they make another common mistake. We illustrate it with the start of a "proof" that the square of any odd integer is odd:

> **Example 3.** *Proof:* Suppose n is any odd integer. By definition of odd, $n = 2k + 1$ for any integer k…

An effective response to the student who starts the proof in this way is to point out that, for example, k cannot be just *any* integer because, in fact, it actually equals $(n - 1)/2$.

Arsac and Durand-Guerrier give an example of a common "proof" of Cauchy's mean value theorem that is erroneous because it fails to observe the dependence rule. They also point out that both Cauchy and Abel occasionally made similar mistakes in their own work. Example 1 shows, however, that this type of error is not limited to advanced proofs but can actually occur in very simple ones.

UNIVERSAL INSTANTIATION AND ITS USE WITH EXISTENTIAL INSTANTIATION

In an elementary proofs course, the following is a typical student's "proof" of the statement: If f is any surjective function from X to Y and g is any surjective function from Y to Z, then the composition $g{\circ}f$ is surjective.

> **Example 4.** *Proof:* By definition of surjective, given any y in Y, there is an x in X with $f(x) = y$. Also by definition of surjective, given any z in Z, there is a y in Y with $g(y) = z$. So $g{\circ}f(x) = g(f(x)) = g(y) = z$.

Of course, the main problem with this proof is that it is backwards. The only way to prove that $g{\circ}f$ is surjective – and the point to emphasize with students – is

to start with a generic element of *Z* and show that there is an element of *X* whose image is that element of *Z*. But, as with the previous examples, we may ask: What leads a student to develop this "proof"?

One possibility is the tendency, noted previously, to regard variables as having a continuing existence beyond the bounds set by the quantification. Thus, the *y* in the first sentence of Example 4 is simply regarded as the same as the *y* in the second sentence.

To analyze Example 4 more deeply, we need to state another principle of logic.

Universal Instantiation: If a property is true for all elements of a set, then it is true for any particular element of the set.

In the first sentence of Example 4 both universal instantiation and existential instantiation are used implicitly in the sense that the naming of *x* and *y* is obviously considered sufficient to allow one to discuss them as if they were particular objects with the property that $f(x) = y$. Similarly, in the second sentence, there is implicit instantiation of both *z* and *y*. The problem, of course, is that the instantiated *y* in the first sentence is generic – it could be any element of *Y* – whereas the instantiated value of *y* in the second sentence depends on the instantiated value of *z*. In other words, the general property of surjectivity stated in the first sentence has to be applied to the particular instantiated value of *y* that is obtained using the second sentence. Copi (1954) pointed out that in general "whenever we use both EI (Existential Instantiation) and UI (Universal Instantiation) in a proof to instantiate with respect to the same individual constant, we must use EI first."

A CAUTIONARY EXAMPLE

Because we are aware of the tendency to invest letters with continuing existence beyond the scope of the quantifier in the sentence where they are introduced, it is common to introduce differentiated symbols even when logic does not actually require them. For example, Selden and Selden (to appear) analyze a proof that the sum of continuous functions is continuous. The middle of the proof contains the following sentences:

[1] Now because *f* is continuous at *a*, there is a $\delta_1 > 0$ such that for any x_1, if $|x_1 - a| < \delta_1$ then $|f(x_1) - f(a)| < \varepsilon/2$.

[2] Also because *g* is continuous at *a*, there is a $\delta_2 > 0$ such that for any x_2, if $|x_2 - a| < \delta_2$ then $|g(x_2) - g(a)| < \varepsilon/2$.

Strictly speaking, the letter *x* could replace both x_1 and x_2 in [1] and [2] because the scopes of the universal quantifiers for x_1 and x_2 only extend to the ends of sentences [1] and [2] respectively. By writing the proof statements in the form shown, the Seldens apparently wanted to avoid any possible misunderstanding that use of a common symbol might induce.

Such well-meaning attempts to solve one problem can, however, occasionally produce another. For example, on an examination, I gave the following problem:

Let $A = \{n \in Z \mid n = 8r - 3$ for some integer $r\}$ and let

$B = \{m \in Z \mid m = 4s + 1$ for some integer $s\}$. Prove that $A \subseteq B$.

In class I had solved a similar problem, starting with a generic element of A and going through the computations needed to show that the element was in B. In the examination problem, I used the letters n and r in the definition for A and the letters m and s in the definition for B to ensure that students would not confound the meanings of the variables in their answers. As things turned out, this effort appears simply to have increased the likelihood of their making a different mistake. Example 5 shows a type of answer made by a large number of students.

Example 5. *Proof:* Let $s = 2r - 1$ for some integer r. Then

$$\begin{aligned} m &= 4(2r - 1) + 1 \quad &&\text{by substitution} \\ &= 8r - 4 + 1 \\ &= 8r - 3 \quad &&\text{by algebra.} \end{aligned}$$

Thus $m = 8r - 3 = n$, hence $m = n$.

Scratch work:
$4s + 1 = 8r - 3$
$4s = 8r - 4$
$s = 2r - 1$

Therefore every element in A is also an element of B and hence $A \subseteq B$.

The proof in Example 5 suggests that although the students had conscientiously learned the details of the computations, they did not fully appreciate the underlying logic of the proof well enough to realize the need to start with a particular but arbitrarily chosen element of A and to show that this element is in B. It is likely that by using different names for the variables in the definitions of A and B, I actually encouraged students to think of them as having an independent existence to which they were entitled to refer without reintroducing instantiations of them in the broader context of the definition of subset.

CONCLUSION

The University of Washington professor Ramesh Gangolli (1991) once made a statement that neatly summarizes an important insight into mathematics instruction:

The mathematics profession as a whole has seriously underestimated the difficulty of teaching mathematics.

The preceding examples illustrate the complexity of some of the logical issues that arise even in simple mathematical proofs. Coming to understand them provides ways for teachers to respond more effectively to students' difficulties.

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