13th International Congress on Mathematical Education
Hamburg, 24-31 July 2016

DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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This paper explores some of the issues involved in teaching a discrete mathematics course for computer science students. The recommendations for discrete mathematics instruction, published by the computer science societies, indicate that these students need to be able to work with mathematical concepts at a relatively sophisticated level. This paper discusses some of the challenges involved in implementing the recommendations, explores some of the reasons behind them, and suggests considerations that educators should take into account when they prepare instructional materials.

INTRODUCTION

Starting with the publication of *Curriculum 1968* by the Association for Computing Machinery (ACM), discrete mathematics became an important part of the recommended program for students of computer science. The current curricular recommendations, prepared by The Joint Task Force on Computing Curricula (2013) of the ACM and the IEEE Computer Society gives discrete mathematics as one of the two largest components (41 instructional hours) in the "core body of knowledge" recommended for all computer science students. The specific topics recommended in the report are sets, relations, and functions; basic logic; proof techniques; basics of counting; graphs and trees; and discrete probability. Details for each topic are given, but the report also indicates that ways of presenting the content may vary considerably from one institution to another. In addition, although the report refers to the subject as "discrete structures," many of the cited textbooks and course titles refer to "discrete mathematics."

In describing the rationale for including the topics, the report states, "The material in discrete structures is pervasive in the areas of data structures and algorithms but appears elsewhere in computer science as well. For example, an ability to create and understand a proof—either a formal symbolic proof or a less formal but still mathematically rigorous argument—is important in virtually every area of computer science, including (to name just a few) formal specification, verification, databases, and cryptography. Graph theory concepts are used in networks, operating systems, and compilers. Set theory concepts are used in software engineering and in databases. Probability theory is used in intelligent systems, networking, and a number of computing applications."

While teaching the course over the past several decades, interaction with students has made me aware that the kinds of logical reasoning that are as natural and unconscious as breathing to me — as a mathematician — are so foreign to many students that they do not easily incorporate them into their own mental processes. Yet these modes of reasoning are necessary for engaging in the kind of abstract thinking that is important for the course and its applications in computer science.

This paper explores three reasons for the problem. One is that there has been a reduced emphasis on mathematical proof in the secondary mathematics curriculum. A second is that the ways certain words and phrases are interpreted in ordinary speech often differ from the ways they must be interpreted in formal, deductive reasoning. A third is the difficulties students experience in understanding and using statements involving variables, both in connection with predicate logic and in interpreting certain kinds of mathematical phrases.

PROOF IN THE SECONDARY MATHEMATICS CURRICULUM

At least in the United States, many students entering colleges and universities have little experience with formal mathematical deduction. Various reasons for this phenomenon have been proposed. Among these are an increase in the population of secondary students taking an academic curriculum, pressures on teachers to prepare students for large-scale tests that do not include mathematical proof, and a de-emphasis on proof in mathematics textbooks. A study of teacher attitudes by Knuth (2002) indicated that the teachers "viewed proof as appropriate for the mathematics education of a minority of students" and that they saw proof "as a topic of study rather than as a tool for communicating and studying mathematics." In a large-scale study of secondary teachers, Frasier and Panasuk (2013) reported that while Knuth's "study gave the mathematics education community reason to be concerned that certain beliefs and perceptions about proof in secondary school mathematics were likely to exist within the population of in-service secondary school mathematics teachers, [our] study gives the mathematics education community reason to be alarmed that these beliefs and perceptions appear to be widespread." They also cited their "analysis of the current Geometry textbooks used in the classrooms" as supporting Knuth's finding about the reduction of proof to a single curricular topic, claiming that "reducing proof to a topic undermines its value as an essential element of mathematics and holds back an opportunity to facilitate the development of logical reasoning and abstract thinking in students."

DIFFERENCES BETWEEN FORMAL AND INFORMAL DISCOURSE

According to Wells (2003) "mathematicians speak and write in a special 'register',,, suited for communicating mathematical arguments…[This] register uses special technical words, as well as ordinary words, phrases and grammatical constructions with special meanings that may be different from their meaning in ordinary English. … Students have various other interpretations of particular constructions used in the mathematical register. One of their tasks as students is to learn how to extract the standard interpretation from what is said and written. One of the tasks of instructors is to teach them how to do that."

One example is that in everyday speech *if-then* is frequently interpreted as *if-and-only-if*. Consider a parent who tells his child, "If you don't eat your dinner then you won't get dessert." If his child ate the dinner and was denied dessert we would certainly condemn the parent even though he only told the child what would happen if the child didn't eat dinner.

Only-if is also routinely interpreted as *if-and-only-if*. We would similarly condemn a parent who tells his child, "You will get dessert only if you eat your dinner" and does not allow the child to have dessert. It is interesting that in *A Dictionary of Modern Legal Usage* Garner (1995) wrote that replacing *only-if* by *if-and-only-if* "is inferior and adds nothing but unnecessary emphasis to *only if*." Perhaps in legal writing the bi-directional *if-and-only-if* is viewed as unnecessary because conditional statements often refer to obligations or requirements and thus only the *only-if* direction is viewed as important while the *if* direction is seen as trivial or irrelevant.

A second example concerns negating a conditional statement. In propositional logic the negation of "if A then B" is the conjunction "A and not B," but many students instinctively write "if A then not-B" instead. And this instinct is so deeply ingrained that I have had difficulty in getting students to resist it. The reason may be that in ordinary speech people often express the negation of "if A then B" as what might be called the "strong negation": "if A then not-B." For instance, a natural response to "If John comes, then the party will be a disaster" is "Don't worry. If John comes, then the party will not be a disaster. It will be fine." Even in certain mathematical situations a person taking issue with a conditional statement might reasonably express dissent by stating the "strong negation" because when the "strong negation" is true, it can seem misleading or even deceptive to disagree by giving only the logical negation. For instance, if someone claims that "If G is a graph with n vertices, then the total degree of G is odd", we might counter by saying, "On the contrary, if G is a graph with n vertices then the total degree of G is even."

An interesting example involving negation and quantification concerns a statement of the form "All A are not B." Even though the statement begins "All A are," it is most often understood not as universal but as existential. A famous example is "All that glisters is not gold" (Shakespeare's *Merchant of Venice*), which means that there are things that glister and are not gold. But examples can be found almost daily in print or broadcast media and even in the mathematics classroom. For example, if a student asks, "Are all graphs planar? " some teachers might answer "No, all graphs are not planar" rather than the completely unambiguous "No, some graphs are not planar."

According to the rules of predicate logic, when a statement contains two different quantifiers, they should be interpreted in left-right order. However in ordinary language a statement with both *for-all* and *there-is* is typically interpreted in whatever way makes the most common sense. Linguists call this "scope ambiguity." For example, the most natural way to interpret the sentence, "There is a lid for every pot" is as "For every pot there is a lid" not as "There is one single lid that will be fit every pot." Similarly, the classic statement, "There is a time for every purpose under heaven" means that every purpose has its own time. A mathematical example is "There is a prime number between every integer greater than 1 and its double."

A somewhat different situation occurs when students are asked to recall a definition involving a conditional statement. For instance, if asked to finish the statement "A is a subset of B if, and only if, for all x in A, ____," students often write "x is in A and x is in B" instead of "if x is in A and x is in B." The same mistake happens when one asks students to write what it means for a relation R on a set A to be symmetric. A surprisingly common response is "x is related to y and y is related to x" rather than "if x is related to y then y is related to x." One possible reason students do this is that in everyday speech statements of the form "if p then q" and "p and q" are sometimes regarded as equivalent. For example, the sentences "If you eat your dinner then you'll get dessert" and "Eat your dinner and you'll get dessert" both promise the same outcome under the same set of circumstances. It is also the case that to evaluate the truth or falsity of a conditional statement, one assumes that the antecedent is true in which case the truth values of "if p then q" and "p and q" are identical.

INTERPRETING AND USING STATEMENTS THAT CONTAIN VARIABLES

In working with discrete mathematics students over the years, I have become increasingly aware of problems they experience when they need to use variables along with the rules of inference for predicate logic.

Universal Generalization and Existential Instantiation

Of the four predicate inference rules, the one that is essentially unique to mathematical thought is universal generalization:

> *Universal generalization*: If we can prove that a property is true for a particular, but arbitrarily chosen, element of a set, then we can conclude that the property is true for every element of the set.

As Alfred North Whitehead (1911) wrote: "Mathematics, as a science, commenced when first someone, probably a Greek, proved propositions about 'any' things or about 'some' things without specification of definite particular things." Because universal generalization is not used in everyday reasoning situations it is not at all obvious to students, even though it may seem like common sense to a mathematician. So I have found it to be worth discussing explicitly when it is needed for a mathematical situation. In presenting the rule to students I often call it "generalizing from the generic particular" because I believe that these words convey a more precise idea than the formal name for how the rule is used in practice.

Usually when universal generalization is used to justify a mathematical statement, it is necessary to introduce a variable or variables to give a name to the objects under consideration. In mathematical situations we can use existential instantiation to do this:

> *Existential Instantiation*: If we know or hypothesize that an object exists, then we may give it a name, as long as we are not using the name for another object in our current discussion.

In mathematical proof-writing, for instance, if we want to prove that the square of any odd integer is odd, universal generalization provides the outline for a proof: namely, that we should start by assuming we have a particular but arbitrarily chosen (or generic) odd integer and then show that its square is odd. Now, because the beginning of the proof assumes that there is such an object, we can use existential instantiation to give it a name, i.e., to introduce a variable to stand for it. And it is the introduction of this variable that enables us to do the calculations that are the heart of the proof.

When I introduce students to proof using universal generalization, I try to lead them to appreciate its extraordinary power: it enables us to deduce a conclusion about all the objects in an infinite set, such as the set of all odd integers, by deducing it for just one, individual element of the set, as long as we make sure we have not assumed anything about the element that is not equally true for every other element of the set.

A significant problem in teaching such a proof is that many students are unaccustomed to working with variables in the ways that mathematicians take for granted, namely as convenient placeholders either for arbitrarily chosen objects or for ones that are fixed but unknown. In theory, because of their work with mathematical word problems in secondary school, students should be used to introducing a variable to stand for an unknown quantity, but even at the tertiary level this is often not an automatic reflex. Jean Dieudonné (1972) once referred to the "boldness" of operating with an unknown "*as if it were a known quantity*," stating that "a modern mathematician is so used to this kind of reasoning that his boldness is now barely perceptible to him." Perhaps given the "boldness" of this reasoning it is understandable that students hesitate to use it.

A consequence of the fact that students perceive variables differently from mathematicians is that they sometimes seem to think of the ones that occur in definitions or theorems as if they continue to exist outside the definitions or theorem where they were introduced. For example, in the early weeks of one discrete mathematics course the following definitions for even and odd integers were given:

An integer is *even* if, and only if, it equals $2k$ for some integer k.

An integer is *odd* if, and only if, it equals $2k + 1$ for some integer k.

As shown in Table 1, when students were asked to prove that the difference of any odd integer minus any even integers is odd, close to 30% wrote something like the following:

$$m - n = (2k + 1) - 2k.$$

Outcome	**Frequency**	**Relative Frequency**
100% correct	12	17%
Small mistakes or incorrect use of language	17	24%
Basic idea with moderate mistakes	7	10%
Got $(2r + 1) - 2s$ then stymied	8	11%
$(2k + 1) - 2k$ mistake	20 + 2 halves	29%
Proof by example	2 + 2 halves	4%
General confusion	4	6%

Table 1: 62 students in two sections of a discrete mathematics class

One way to look at the $(2k + 1) - 2k$ mistake is that it involves a misapplication of existential instantiation because the second use of the letter k violates the requirement that in giving an object a name we cannot use one that has previously been introduced for a different object in our discussion. A related but somewhat different way to think of it is that it violates what Durand-Guerrier and Arsac refer to as the "dependence rule": namely, that in the sentence "For all x in set D, there exists a y in set E such that…" the value of y depends on the value of x.

To help students understand the mistake, one can ask, "How are numbers $2k$ and $2k + 1$ related?" "If $m = 2k$ and $n = 2k + 1$, what is the relation between m and n?" "Does your argument show that the difference between *any* odd integer and *any* even integer is odd?"

A more complex example of a failure to observe the dependence rule is the following:

Theorem: If X, Y, and Z are sets and $f : X \rightarrow Y$ and g: $Y \rightarrow Z$ are both onto functions, then $g \circ f : X \rightarrow Z$ is onto.

"*Proof:* Suppose $f : X \rightarrow Y$ and g: $Y \rightarrow Z$ are both onto. Because f is onto, for all y in Y there exists an x in X such that $y = f(x)$, and because g is onto, for all z in Z there exists y in Y such that $g(y) = z$. Thus $g(f(x)) = g(y) = z$ and so the composition $g \circ f$ is onto."

Mathematical usage may actually encourage students to imagine that the variable k has meaning that persists outside of the definitions in which it appears. When variables are used in the universal quantification for a theorem statement, mathematicians often start a proof by using the same variable names as if they had been introduced as arbitrary elements satisfying the hypothesis. Here is an example.

Theorem: For all integers m and n, if m is odd and n is even, then $m - n$ is odd.

Proof: Since m is even and n is odd, there are integers r and s such that $m = 2r + 1$ and $n = 2s + 1$. Therefore, $m - n = (2r + 1) - 2s = \ldots$ etc.

Universal Instantiation

Another challenge for students is the use of variables in connection with the predicate inference rule known as universal instantiation:

Universal instantiation: If a property is true for all elements of a set, then it is true for each individual element of the set.

For example, when a property, such as the distributive law for sets is given as

For all sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

students often fail to see that that they can to use the law (by universal instantiation) to transform an expression like $(A \cap B) \cap (B^c \cup A)$ into $[(A \cap B) \cap B^c] \cup [(A \cap B) \cap A]$, en route to showing that it equals $A \cap B$. Similarly, when Fibonacci numbers F_0, F_1, $F_2, \ldots$ are defined by specifying that

$$F_0 = 1,\ F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for all integers } n > 1,$$

many students struggle to write an expression for F_{n+1}. Finally consider a recursive definition, such as the one for a set of Boolean expressions specifying that all the letters of the English alphabet – a, b, c, … – are Boolean expressions and that if P and Q are Boolean expressions, then so are

(a) $(P \wedge Q)$ and (b) $(P \vee Q)$ and (c) $\sim P$.

Yet, when given this definition, it is common for students to be at a loss when asked to show that, say, $(\sim q \wedge r)$ is a Boolean expression, even when they have previously seen examples that are similar in that they involve more than one step.

In each case it is helpful for students to come to perceive the variables used to express a law or definition as arbitrary symbols that simply hold places into which other values can be substituted. To make the point, one can state the properties several times using different variable names each time. One can even put empty boxes in place of the variable names in order to create a template of explicit placeholders. For example, to express the distributive law for sets, one can write that

$\square \cap (\triangle \cup \bigcirc) = (\square \cap (\triangle) (\square \cup \bigcirc)$ no matter what sets are put into the boxes, explaining (for the example above) that one can apply the distributive law by putting ($A \cap B$) in the ☐ boxes, B^c in the △ boxes, and A in the ○ boxes. Similarly for the definition of a recurrence relation, such as the Fibonacci numbers, one can write

$$F_0 = 1,\ F_1 = 1, \text{ and } F_{\triangle} = F_{\triangle-1} + F_{\triangle-1} \text{ for all integers greater than 1 placed into the box.}$$

And to express the definition for a set of Boolean expressions one can write that all the letters of the English alphabet – a, b, c, … – are Boolean expressions and that when any Boolean expression is placed into the ☐ box or the △ box below, the result is also a Boolean expression:

$$\text{(a) } (\square \wedge \triangle) \quad \text{and} \quad \text{(b) } (\square \vee \triangle) \quad \text{and} \quad \text{(c) } \sim\square\,.$$

For Boolean expressions it is particularly important to emphasize the multi-step nature of the derivations.

Existential Generalization

For completeness I include a statement of the fourth inference rule for the predicate calculus:

> *Existential Generalization*: If we know that a certain property is true for a particular object, then we may conclude that there exists an object for which the property is true.

Of course this rule is very important in mathematical reasoning because it is the logical justification for disproving a universal statement by supplying a counterexample to show the truth of its negation. But existential generalization does not seem to be a problem for students. It is often applied virtually without reference to variables and with the final existential statement regarded as so intuitively obvious that it is not even stated explicitly. For example given the statement “All prime numbers are odd,” a typical disproof would simply look something like the following: “Counterexample: The number 2 is a prime number and 2 is not odd.”

Linguistic Abbreviations in Mathematics

Much mathematical language is expressed in a kind of slang, where success depends on recognizing the meaning behind what is actually written or spoken. Table 2 summarizes some of the disparities between what we say and what we mean when we use linguistic expressions involving variables. Some students do not seem to need explicit instruction to learn how to interpret the meanings underlying the phrases, but for other students the expressions do not necessarily evoke the understanding that an instructor or textbook may assume.

What we say	**What we mean**
the value of x	the quantity that is put in place of x
as the value of x increases	as larger and larger numbers are put in place of x
As the value of x increases, the value of y increases.	If larger and larger numbers are put in place of x, the corresponding numbers that are put in place of y become larger and larger.
where x is any real number	for all possible substitutions of real numbers in place of x

Let *n* be any even integer.	Imagine substituting an integer in place of *n* but do not assume anything about its value except that it is an even integer.
By definition of even, $n = 2k$ for some integer *k*.	By definition of even, there is an integer we can substitute in place of *k* so that the equation $n = 2k$ will be true. (In fact there is only one such integer; its value is $n/2$.)
the function x^2	the function that relates each real number to the square of that number. In other words, for each possible substitution of a real number in place of *x*, the function corresponds the square of that number.
where *x* is some real number that satisfies the given property	There is a real number that will make the given property true if we substitute it in place of *x*.

Table 2: What we say vs. what we mean

FINAL THOUGHTS

This paper has focused on the special challenges for instructors teaching a discrete mathematics course for students of computer science. In much of the 20th century mathematical pedagogy could focus on the small set of students who were able to absorb advanced mathematical thought processes from textbooks and instructors seemingly by osmosis. In the world of today, however, there is a need for a larger fraction of the population to be able to work with mathematical concepts at a relatively sophisticated level, and this may especially be the case for students wishing to concentrate in informatics/computer science. This paper has tried to address considerations for educators to take into account when they prepare instruction for such students.

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