This help document accompanies Richard Johnsonbaugh: *Discrete Mathematics*, 7th edition, Prentice Hall, Upper Saddle River, N.J., 2009.

Equivalent Characterizations of Trees

A tree is, by Definition 9.1.1, a simple graph satisfying the following: If v and w are vertices in T, there is a simple path from v to w. According to Theorem 9.2.3, if T is a graph with n vertices this definition is equivalent to

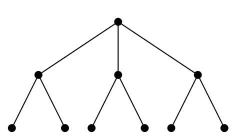
(i) T is connected and acyclic.

(ii) T is connected and has $n-1$ edges.

(iii) T is acyclic and has $n-1$ edges.

Recall that *acyclic* means "no cycles."

Example. Consider the graph



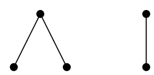
This graph is simple, and it satisfies the following: If v and w are vertices in T, there is a unique simple path from v to w. Therefore this graph is a tree. The graph is connected and acyclic and, therefore, satisfies property (i). The graph is connected and has 10 vertices and nine edges and, therefore, also satisfies property (ii). The graph is acyclic and has 10 vertices and nine edges and, therefore, also satisfies property (iii). □

Example. Explain why there is no tree having 10 vertices with degrees 1,1,1,1,2,2,2,2,2,2.

Solution. Suppose by way of contradiction that such a tree exists. The sum of the degrees is 16. By Theorem 8.2.21, there must be $16/2 = 8$ edges. But a tree with 10 vertices must have nine edges. This contradicts properties (ii) and (iii). Therefore, no such tree exists. □

Example. Give an example of an acyclic graph that is not a tree.

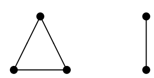
Solution. The graph



is acyclic, but it is not a tree. Notice that it is not connected, and that is has five vertices and three edges. □

Example. Give an example of a graph that has five vertices and four edges but is not a tree.

Solution. The graph



has five vertices and four edges but it is not a tree; it contains a cycle and it is not connected. □

Example. Suppose that we begin with a graph consisting of no edges and one vertex. We then repeatedly choose a vertex v in the existing graph and add a vertex w and edge (v, w). Show that at each iteration, the existing graph is a tree.

Solution. Since we always add an edge incident on a vertex in the existing graph, at each iteration the graph is connected. Since the added edge is always incident on a new vertex, the graph never contains a cycle. Therefore the graph is always connected and acyclic. Thus, at each iteration, the existing graph is a tree. □

Example. Suppose that a graph G has n vertices and $n-1$ edges and is not a tree. Show that G is not connected and contains a cycle.

Solution. We argue by contradiction. Negating the conclusion (using De Morgan's Law) gives: G is connected *or* G is acyclic. If G is connected, by property (ii) G is a tree. Contradiction. If G is acyclic, by property (iii) G is a tree. Contradiction. □