WebHelp: Mathematical Induction

Mathematical induction is used to prove a sequence of statements indexed by the positive integers. For example, if

\[ S(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \]

mathematical induction can be used to prove that \( S(n) \) is true for all positive integers \( n \). In other words, mathematical induction can be used to prove that

\[
\begin{align*}
S(1) & : 1 = \frac{1 \cdot 2}{2}, \\
S(2) & : 1 + 2 = \frac{2 \cdot 3}{2}, \\
S(3) & : 1 + 2 + 3 = \frac{3 \cdot 4}{2}, \\
& \vdots \\
S(n) & : 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \\
S(n+1) & : 1 + 2 + 3 + \cdots + n + (n + 1) = \frac{(n + 1)(n + 2)}{2}, \\
& \vdots
\end{align*}
\]

are all true. [Notice that \( S(n+1) \) is obtained from \( S(n) \) by everywhere substituting \( n + 1 \) for \( n \).]

In its simplest form (we discuss the strong form of mathematical induction at the end of this WebHelp), mathematical induction requires two steps

- **Basis Step.** Prove that \( S(1) \) is true.
• **Inductive Step.** For every $n$, *assume* that $S(n)$ is true and *prove* that $S(n+1)$ is true.

To see why the Basis and Inductive Steps prove that $S(n)$ is true for all $n$, consider any specific value of $n$, for example, $n = 5$. Is $S(5)$ true? Well, because of the Basis Step, $S(1)$ is true. The Inductive Step says that if $S(1)$ is true, then $S(2)$ is true. $S(1)$ is true! Therefore $S(2)$ is true. The Inductive Step says that if $S(2)$ is true, then $S(3)$ is true. $S(2)$ is true! Therefore $S(3)$ is true. The Inductive Step says that if $S(3)$ is true, then $S(4)$ is true. $S(3)$ is true! Therefore $S(4)$ is true. The Inductive Step says that if $S(4)$ is true, then $S(5)$ is true. $S(4)$ is true! Therefore $S(5)$ is true! We could use a similar argument for any value of $n$, therefore $S(n)$ is true for every $n$.

The Basis Step is usually straightforward. In the previous example, the Basis Step is to prove

$$1 = \frac{1 \cdot 2}{2},$$

which is certainly true! In the previous example, the Inductive Step is to *assume* that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

is true, and then prove that

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

is true. *We recommend that you always write out both case $n$ and case $n+1$ before proceeding further with the Inductive Step.*

The key to proving the Inductive Step is to “uncover” case $n$ within case $n+1$. Although the meaning of “uncover” depends on the context, it is *always* the case that the success of the Inductive Step rests on uncovering case $n$ within case $n+1$.

For our example, case $n$ involves

$$1 + 2 + \cdots + n,$$

which appears within case $n+1$:

$$1 + 2 + \cdots + n + (n+1).$$

This is case $n$.

Since we are assuming that case $n$ is true, that is, that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

we may substitute

$$\frac{n(n+1)}{2}$$
for
\[1 + 2 + \cdots + n\]
in case \(n + 1\) to obtain
\[1 + 2 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1)\).

[Notice that the term \((n + 1)\) was not replaced by anything and, so, is just copied from the left side to the right side of the equation.] The Inductive Step is completed by using algebra to get the right side into the correct form, namely,
\[\frac{(n + 1)(n + 2)}{2}.
\]

We have
\[1 + 2 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1)\]
\[\quad \text{factor out } n + 1\]
\[\quad = (n + 1) \left(\frac{n}{2} + 1\right)\]
\[\quad = \frac{(n + 1)(n + 2)}{2}.
\]

The Inductive Step is finished and the proof by mathematical induction is complete.

**Summary**

To give a proof that \(S(n)\) is true for every positive integer \(n\) using mathematical induction in its simplest form:

- Prove directly that \(S(1)\) is true.

- Assume that \(S(n)\) is true and prove that \(S(n + 1)\) is true. As an aid, write out \(S(n)\) and \(S(n + 1)\) explicitly, remembering that \(S(n + 1)\) is obtained from \(S(n)\) by everywhere replacing \(n\) by \(n + 1\).

To help with the latter step, look for \(S(n)\) within \(S(n + 1)\).

**Strong Form of Mathematical Induction**

In the strong form of mathematical induction, the preceding Inductive Step

- **Inductive Step.** For every \(n\), assume that \(S(n)\) is true and prove that \(S(n + 1)\) is true.

is replaced by

- **Inductive Step for Strong Form of Mathematical Induction.** For every \(n\), assume that \(S(k)\) is true for all \(k < n + 1\) and prove that \(S(n + 1)\) is true.
The Basis Step is unchanged. In the strong form of mathematical induction, to prove that $S(n + 1)$ is true we may assume the truth of $S(k)$ for all $k$ that precede $n + 1$, namely $1, 2, \ldots, n$. In the simple form of mathematical induction, to prove that $S(n+1)$ is true, we assume the truth of $S(k)$ only for the $k$ that immediately precedes $n+1$, namely $n$. In other words, in the strong form of mathematical induction, to prove that $S(n + 1)$ is true, we assume that $S(1), S(2), \ldots, S(n)$ are all true. In the simple form of mathematical induction, to prove that $S(n + 1)$ is true, we assume only that $S(n)$ is true. The strong form of mathematical induction is useful when the truth of all the preceding cases helps prove case $n+1$. Example 1.6.7 in the book illustrates.