1. Introduction

In [11], Shimura discovered the following remarkable connection between holomorphic eigenforms of half-integral weight and their integral weight counterparts.

Theorem 1.1. — Let $F$ be a nonzero holomorphic cusp form of level $4N$, weight $k/2 \geq 5/2$ and nebentypus $\chi$. Assume that $F$ is an eigenform for $T_\ell^2$ and $U_\ell^2$ for all primes $\ell$ with eigenvalues $\alpha_\ell$. Then there exists a nonzero holomorphic cusp form $f$ of weight $k-1$, level $2N$, and character $\chi^2$ that is an eigenform for all $T_\ell$ and $U_\ell$ with eigenvalues $\alpha_\ell$.

Strictly speaking, Shimura had only conjectured that $f$ is of level $2N$, but this was proven shortly thereafter by Niwa in [8] for weights at least $7/2$ and then by Cipra for all weights at least $5/2$ in [3].

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Let $p$ be an odd prime and let $N$ be a positive integer with $p \nmid N$. In [9] the author constructed a rigid analytic space $\tilde{D}$ (denoted $\tilde{D}^0$ there) parameterizing all finite-slope systems of eigenvalues of Hecke operators acting on overconvergent cuspidal $p$-adic modular forms of half-integral weight and tame level $4N$. Let $D$ denote the integral weight cuspidal eigencurve of tame level $2N$ constructed, for example, in [1]. In this paper we construct a rigid-analytic map $\text{Sh} : \tilde{D}_{\text{red}} \rightarrow D_{\text{red}}$ that interpolates the Shimura lifting in the sense that if $x \in \tilde{D}$ is a system of eigenvalues occurring on a classical cusp form $F$ of half-integral weight (and such points are shown to be Zariski-dense in $D$) then $\text{Sh}(x)$ is the system of eigenvalues associated to the classical Shimura lifting of $F$.

2. Modular forms of half-integral weight

Fix an odd prime $p$ and let $W$ denote $p$-adic weight space over $\mathbb{Q}_p$. We briefly recall a few facts and bits of notation concerning $W$. See Section 2.4 of [9] for more details. The $K$-valued points of $W$ (for a complete extension $K/\mathbb{Q}_p$) correspond to continuous characters $\kappa : \mathbb{Z}_p \rightarrow K^\times$. Each $\kappa \in W(K)$ factors uniquely as $\kappa = \tau^i \cdot \kappa'$ where $\tau$ is the Teichmuller character, $i$ is an integer well-defined modulo $p-1$, and $\kappa'$ is trivial on $\mu_{p-1} \subseteq \mathbb{Z}_p^\times$. The space $W$ is accordingly the admissible disjoint union of $p-1$ subspaces $W^i$ for $0 \leq i < p-1$. Each $W^i$ is isomorphic to the open unit ball $B(1,1)$ around 1 under the map

$$\kappa \mapsto \kappa(1+p).$$

Also, $W$ is the rising union of the nested sequence of admissible affineoids $\{W_n\}$ whose points are those $\kappa$ with $|\kappa(1+p)^{p^n-1} - 1| \leq |p|$. For an integer $i$ with $0 \leq i < p-1$ and a positive integer $n$ we set $W^i_n = W^i \cap W_n$. Finally, if $\lambda \geq 0$ is an integer, we will usually denote the $\mathbb{Q}_p$-valued point $x \mapsto x^\lambda$ of $W$ simply by $\lambda$.

Let $N$ be a positive integer not divisible by the odd prime $p$. Given a $p$-adic weight $\kappa \in W(K)$, with $K$ and complete and discretely-valued extension of $\mathbb{Q}_p$, and an $r \in [0, n_r] \cap \mathbb{Q}$, we introduced in [9] the Banach space $\tilde{M}_\kappa(4N, K, p^{-r})$ of half-integral weight $p$-adic modular forms of tame level $4N$ and weight $\kappa$ defined over $K$. Here $\{n_r\}$ is the decreasing sequence of positive rational numbers introduced in [1] and [9], the details of which are to be found in Section 3. This space is endowed with a continuous action of the Hecke operators $T_{d \ell^2}$ ($\ell \nmid 4Np$) and $U_{d \ell^2}$ ($\ell \nmid 4Np$), as well as the tame diamond operators $(d)_{4N}$, $(d \in (\mathbb{Z}/4N\mathbb{Z})^\times)$). A cuspidal subspace $\tilde{S}_\kappa(4N, K, p^{-r})$ is also defined, and is equipped with all of the same operators. In this section we will define a space of classical modular forms of half-integral weight and use it and results of [9] to define the classical subspaces of $\tilde{M}_\kappa(4N, K, p^{-r})$ and its cuspidal analog. We will then recast the classical Shimura lifting (Theorem 1.1) in these terms.

Remark 2.1. — The classical subspaces considered in this paper are limited in the sense that we restrict our attention to classical forms of level $4Np$. One can also include classical forms of higher level $4Np^m$ into the above spaces of $p$-adic forms.
We omit these forms here in part because we have no real need for them, and in part because we have not proven an analog of our control theorem (Theorem 2.4) for such forms (though we expect such a result to hold).

For any positive integer $M$, let $\Sigma_{4M}$ be the $Q$-divisor on the algebraic curve $X_1(4M)_Q$ given by
\[
\Sigma_{4M} = \frac{1}{4} \pi^* [c]
\]
where $c$ is the cusp on (the coarse moduli scheme) $X_1(4)_Q$ corresponding to the pair $(\text{Tate}(q), \zeta_{4q^2})$ and \[\pi : X_1(4M)_Q \to X_1(4)_Q\]
is the natural map. This divisor $\Sigma_{4M}$ is set up to look like the divisor of zeros of the pullback of the Jacobi theta-function $\theta_k$ to $X_1(4M)_Q$. Indeed, if $F$ is a meromorphic function on $X_1(4M)_Q$, then $F\theta^k$ is a holomorphic modular form of weight $k/2$ if and only if $\text{div}(F) \geq -k\Sigma_{4M}$.

Let $C_{4M}$ be the divisor on $X_1(4M)_Q$ given by the sum of the cusps at which $\Sigma_{4M}$ has integral coefficients (this includes, in particular, all cusps outside of the support of $\Sigma_{4M}$). If $F$ is a meromorphic function on $X_1(4M)_Q$, then $F\theta^k$ is a cuspidal modular form of weight $k/2$ if and only if $\text{div}(F) \geq -k\Sigma_{4M} + C_{4M}$. The reason for omitting the cusps at which $\Sigma_{4M}$ has non-integral coefficients is that, since $\text{div}(F)$ has integral coefficients, $F\theta^k$ automatically vanishes at such a cusp as soon as it is holomorphic there.

**Definition 2.2.** — Let $k$ be an odd positive integer. The space of classical modular forms of weight $k/2$ and level $4M$ over $K$ is defined by
\[
\tilde{M}^{cl}_{k/2}(4M, K) = H^0(X_1(4M)_K^\text{an}, \mathcal{O}(k\Sigma_{4M})),
\]
and the subspace of cusp forms is defined by
\[
\tilde{S}^{cl}_{k/2}(4M, K) = H^0(X_1(4M)_K^\text{an}, \mathcal{O}(k\Sigma_{4M} - C_{4M})).
\]
Both of these spaces are endowed with a geometrically defined action of the Hecke operators $T_\ell (\ell \nmid 4M)$ and $U_\ell (\ell \mid 4M)$ as well as the diamond operators $(d)$ ($d \in (\mathbb{Z}/4N\mathbb{Z})^\times$). The construction of the Hecke operators is a “twisted” version of the usual pull-back/push-forward through the Hecke correspondence where one must multiply by a well-chosen rational function (essentially the ratio of the pull-backs of $\theta^k$ through the maps defining the correspondence) on the source space of the correspondence. This construction is carried out in Section 6 of [10] and Section 5 of [9] (where it is applied to a slightly different space of forms). The diamond operators are simply given by pull-back with respect to the corresponding automorphisms of $X_1(4M)_Q$.

Suppose now that $M = Np$ with $p$ and odd prime not dividing $N$. For reasons of $p$-adic weight character book-keeping we separate the diamond action into two kinds of diamond operators using the Chinese remainder theorem. For $d \in (\mathbb{Z}/p\mathbb{Z})^\times$ we define $d_p$ to be $d'$ where $d'$ is chosen so that $d' \equiv d \pmod{p}$ and $d' \equiv 1 \pmod{4N}$. The
operators \( \langle d \rangle_{4N} \) for \( d \in (\mathbb{Z}/4NZ)^\times \) are defined similarly, and for any \( d \) prime to \( 4Np \) there is a factorization \( \langle d \rangle = \langle d \rangle_p \circ \langle d \rangle_{4N} \).

Let \( k \) be an odd positive integer and define \( \lambda = (k - 1)/2 \). In Section 6 of [9] we defined an injection
\[
\tilde{M}_{k/2}^{cl}(4Np, K) \longrightarrow \tilde{M}_{\lambda \tau}^{\dagger}(4N, K, p^{-r})
\]
for any rational number \( r \) with \( 0 \leq r \leq r_1 \), where \( \langle \rangle^r \) indicates the eigenspace of the \( j^\text{th} \) power of the Teichmüller character \( \tau \) for the action of the \( \langle d \rangle_p \). This injection is equivariant with respect to all of the Hecke operators and tame diamonds operators \( \langle \rangle_{4N} \). It is also compatible with varying \( r \) and in particular furnishes an injection
\[
\tilde{M}_{k/2}^{cl}(4Np, K) \longrightarrow \tilde{M}_{\lambda \tau}^{\dagger}(4N, K)
\]
into the space of all overconvergent forms. If we further restrict to the eigenspace of a Dirichlet character \( \chi \) modulo \( 4N \) (valued in \( K \)) for the \( \langle d \rangle_{4N} \), then we also get an embedding
\[
\tilde{M}_{k/2}^{cl}(4Np, K, \chi^{\tau^j}) \longrightarrow \tilde{M}_{\lambda \tau}^{\dagger}(4N, K, \chi)
\]
of the space of classical forms with nebentypus character \( \chi^{\tau^j} \) for the entire group of diamond operators \((\mathbb{Z}/4Np\mathbb{Z})^\times\) into the space of overconvergent forms of tame nebentypus \( \chi \). The image of any of these injections will be referred to as the classical subspace of the target. Note that this definition is consistent with the definition of a classical form given in [9].

**Remark 2.3.** — Everything in the previous paragraph goes through verbatim when restricted to the respective spaces of cusp forms. However, we caution that it is possible for a non cuspidal classical form to be mapped to the space of \( p \)-adic cusp forms \( S_{\lambda \tau}^i(4N, K) \) under the above inclusions; the form need only vanish at the cusps in the connected component \( X_1(4Np)_{an}^{\mathbb{Z}_1} \) of the ordinary locus in \( X_1(4Np)_{an}^{K} \). Still, by the classical subspace of \( S_{\lambda \tau}^i(4N, K) \) we mean the image of the map
\[
\tilde{S}_{k/2}^{cl}(4Np, K) \longrightarrow S_{\lambda \tau}^{i}(4N, K).
\]

**Theorem 2.4.** — Let \( F \) be an element of \( \tilde{M}_{\lambda \tau}^{i}(4N, K) \) or \( \tilde{S}_{\lambda \tau}^{i}(4N, K) \) and suppose that there exists a monic polynomial \( P(T) \in K[T] \) all of whose roots have valuation less than \( 2\lambda - 1 \) such that \( P(U_{p^2})F = 0 \). Then \( F \) is classical.

**Proof.** — The case of \( F \in \tilde{M}_{\lambda \tau}^{i}(4N, K) \) was settled in [9] (Theorem 6.1). The proof follows Kassaei’s approach in [7] and builds the classical form by analytic continuation and and gluing. In particular, one writes down an explicit sequence of forms on the “other” component of the ordinary locus of \( X_1(4Np)_{an}^{K} \) that converges to the analytic continuation of \( F \). It is easy to see that these forms vanish at all cusps as long as \( F \) does, so the proof of carries over to the cuspidal case verbatim. 

For later ease of use, we translate Theorem 1.1 into the world of \( p \)-adic coefficients.
**Theorem 2.5.** — Suppose that $k \geq 5$ and $F \in \tilde{S}^{cl}_{k/2}(4Np, K, \chi \tau^j)$ is a nonzero classical eigenform for all Hecke operators $T_{\ell^2}$ and $U_{\ell^2}$ with eigenvalues $\alpha_{\ell} \in K$. Then there exists a unique nonzero normalized classical cuspidal modular form $f$ of weight $k - 1$, level $2Np$, and nebentypus $\tau^j \chi^2$ defined over $K$ that is an eigenform for all Hecke operators $T_{\ell}$ and $U_{\ell}$ with eigenvalues $\alpha_{\ell}$.

**Proof.** — Fix an embedding $i : K \hookrightarrow \mathbb{C}$. By $p$-adic GAGA, any $F \in \tilde{M}^{cl}_{k/2}(4Np, K)$ is the analytification of an element of $H^0(X_{1}(4Np)_{K}, \mathcal{O}(\Sigma_{4Np}))$. Pulling back via the embedding $i$ and passing to the complex analytic space we arrive at an element $F_i \in H^0(X_{1}(4Np)_{\mathbb{C}}, \mathcal{O}(\Sigma_{4Np}))$ depending on $i$. The condition on the divisor of $F_i$ exactly guarantees that the meromorphic modular form $F_i \theta^k$ of weight $k/2$ is in fact holomorphic. Moreover the association $F \mapsto F_i \theta^k$ is equivariant for the action of Hecke and diamond operators on both sides. This can be seen as a formal consequence of the (entirely parallel) construction of these operators on both spaces. Alternatively, in case of the Hecke operators, this can be deduced by examining their effect on $q$-expansions. Replacing the divisor $\Sigma_{4Np}$ with $\Sigma_{4Np} - C_{4Np}$ we see that the association $F \mapsto F_i \theta^k$ also preserves the condition of cuspidality.

Suppose that $F \in \tilde{S}^{cl}_{k/2}(4Np, K, \chi \tau^j)$ satisfies

\[
T_{\ell^2}F = \alpha_{\ell}F \quad \text{for all } \ell \nmid 4Np \\
U_{\ell^2}F = \alpha_{\ell}F \quad \text{for all } \ell | 4Np \\
\langle d \rangle_{4N}F = \chi(d)F \quad \text{for all } d \in (\mathbb{Z}/4N\mathbb{Z})^\times
\]

for some Dirichlet character $\chi \mod 4N$, with $\alpha_{\ell}, \chi(d) \in K$ for all $\ell$ and $d$. It follows that the holomorphic cusp form $F_i \theta^k$ is of weight $k/2$, level $4Np$, nebentypus character $i \circ (\tau^j \chi)$, and is an eigenform for all $T_{\ell^2}$ and $U_{\ell^2}$ with eigenvalues $\alpha_i(\ell)$ by the classical lifting theorem (Theorem 1.1). We can associate to this form a cuspidal modular form $f_i$ of weight $k - 1$, level $2Np$, and nebentypus character $i \circ (\tau^j \chi^2)$ that is an eigenfunction for all $T_{\ell}$ and $U_{\ell}$ with eigenvalues $\alpha_i(\ell)$.

By complex-analytic GAGA, this form is actually an algebraic modular form defined over $\mathbb{C}$ with all the same properties. The $q$-expansion coefficients of $f_i$ at the cusp $(\text{Tate}(q), e^{2\pi i/4Np})$ are the leading coefficient $a_1(f_i)$ times polynomials in the Hecke eigenvalues $\alpha_i(\ell)$. Since $f_i \neq 0$, $a_1(f_i) \neq 0$ as well and we may normalize $f_i$ so that $a_1(f_i) = 1$. It now follows from the $q$-expansion principle that $f_i$ is in fact an algebraic modular form defined over the field $K$ of weight $k - 1$, level $4Np$, and nebentypus $\tau^j \chi^2$ that is an eigenform for all the $T_{\ell}$ and $U_{\ell}$ with eigenvalues $\alpha_{\ell}$. Moreover, $f_i$ is completely determined by the eigenvalues $\alpha_{\ell}$ for all $\ell$ and is therefore unique (and in particular independent of $i$).

\[
\square
\]

### 3. The eigencurves

As the details of the construction of the relevant eigencurves will be used extensively in the sequel, we briefly recall them here. The construction uses various Banach modules of modular forms equipped with a Hecke action. We refer the reader to Sections 6 and 7 of [1] for the integral weight definitions and to Sections 4 and 5
of [9] for the half-integral weight definitions. We also refer the reader to [1] for foundational details concerning the Fredholm theory that goes into the construction of eigenvarieties in general.

For the moment, let \( W \) be any reduced rigid space over a complete and discretely-valued extension field \( K \) of \( \mathbb{Q}_p \). Fix a set \( \mathbf{T} \) with a distinguished element \( \phi \in \mathbf{T} \).

Suppose that we are given, for each admissible affinoid open \( X \subseteq W \), an \( \mathcal{O}(X) \)-Banach module \( M_X \) satisfying property \((Pr)\) of [1], equipped with map

\[
\begin{align*}
\mathbf{T} &\rightarrow \text{End}_{\mathcal{O}(X)}(M_X) \\
t &\mapsto t_X
\end{align*}
\]

whose image consists of commuting continuous endomorphisms and such that \( \phi_X \) is compact for each \( X \). Suppose also that for each pair \( X_1 \subseteq X_2 \subseteq W \) of admissible affinoid opens we are given a continuous injective map \( \alpha_{12} : M_{X_1} \rightarrow \hat{M}_{X_2} \otimes_{\mathcal{O}(X_2)} \mathcal{O}(X_1) \) of \( \mathcal{O}(X_1) \)-modules that is a “link” in the sense of [1]. Finally, suppose that these links commute with \( \mathbf{T} \) in the sense that \( \alpha_{12} \circ t_{X_1} = (t_{X_2} \otimes 1) \circ \alpha_{12} \) for each \( t \in \mathbf{T} \) and that they satisfy the cocycle condition \( \alpha_{13} = \alpha_{23} \circ \alpha_{12} \) for any triple \( X_1 \subseteq X_2 \subseteq X_3 \subseteq W \) of admissible open affinoids.

Out of this data one can use the machinery of [1] to construct rigid analytic spaces \( D \) and \( Z \) over \( K \) called the eigenvariety and spectral variety, respectively, equipped with canonical maps

\[
D \rightarrow Z \rightarrow W.
\]

The points of \( D \) correspond to systems of eigenvalues of \( \mathbf{T} \) acting on the modules \( \{M_X\} \) such that the \( \phi \)-eigenvalue is nonzero, in a sense made precise below in Lemma 3.3, and the map \( D \rightarrow Z \) simply records the reciprocal of the \( \phi \)-eigenvalue and a point in \( W \).

The space \( Z \) is easy to define. For any admissible affinoid \( X \subseteq W \) we define \( Z_X \) to be the zero locus of the Fredholm determinant

\[
P_X(T) = \det(1 - \phi_X T | M_X)
\]

in \( X \times \mathbb{A}^1 \). The links guarantee that this determinant is independent of \( X \) in the sense that if \( X_1 \subseteq X_2 \subseteq W \) are two admissible open affinoids, then \( P_{X_1}(T) \) is the image of \( P_{X_2}(T) \) under the natural restriction map on the coefficients. It follows that we can glue the \( Z_X \) for varying \( X \) covering \( W \) to obtain a space \( Z \) equipped with a map \( Z \rightarrow W \).

The construction of \( D \) is more complicated, and involves first finding a nice admissible cover of \( Z \) and constructing the part of \( D \) over each piece separately and then gluing these pieces together. This cover is furnished by the following theorem (Theorem 4.6 of [1]).

**Theorem 3.1.** — Let \( R \) be a reduced affinoid algebra over \( K \), let \( P(T) \) be a Fredholm series over \( R \), and let \( Z \subset \text{Sp}(R) \times \mathbb{A}^1 \) denote the hypersurface cut out by \( P(T) \) equipped with the projection \( \pi : Z \rightarrow \text{Sp}(R) \). Define \( \mathcal{C}(Z) \) to be the collection of admissible affinoid opens \( Y \) in \( Z \) such that...
The overconvergent Shimura lifting

- \( Y' = \pi(Y) \) is an admissible affinoid open in \( \text{Sp}(R) \),
- \( \pi : Y \rightarrow Y' \) is finite, and
- there exists \( e \in \mathcal{O}(\pi^{-1}(Y')) \) such that \( e^2 = e \) and \( Y \) is the zero locus of \( e \).

Then \( \mathcal{C}(Z) \) is an admissible cover of \( Z \).

We will generally take \( Y' \) to be connected in what follows. This is not a serious restriction, since \( Y \) is the disjoint union of the parts lying over the various connected components of \( Y' \). We also remark that the third of the above conditions follows from the first two (this is observed in [1] where references to the proof are supplied).

Fix an admissible open affinoid \( X \subseteq W \) and fix \( Y \in \mathcal{C}(Z_X) \) with connected image \( Y' \subseteq X \). Let

\[
P_Y(T) = \det(1 - (\phi_X \hat{\otimes} 1)T) \mid M_X \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(Y'))
\]

Note that this is not in conflict with the existing notation \( p_Y \), (for an arbitrary connected admissible affinoid open \( Y' \subseteq W \)) by Lemma 2.13 of [1] and the above comments about the independence of \( P_X \) on \( X \). When an ambient \( X \subseteq W \) is fixed, we prefer to use the definition (1) instead so as to avoid using the links.

As explained in Section 5 of [1], we can associate to the choice of \( Y \) a factorization \( P_Y(T) = Q(T)Q'(T) \) into relatively prime factors with constant term 1, where \( Q \) is a polynomial of degree equal to the degree of the projection \( \pi : Y \rightarrow Y' \) whose leading coefficient is a unit. Geometrically speaking, \( Y \) is the zero locus of the polynomial \( Q \) in \( \pi^{-1}(Y') \) while its complement \( \pi^{-1}(Y') \setminus Y \) is cut out by the Fredholm series \( Q' \).

By the Fredholm theory of [1] there is a unique decomposition

\[
M_X \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(Y') \cong N \oplus F
\]

into closed \( \phi \)-invariant submodules with the property that \( Q^*(\phi) \) vanishes on \( N \) and is invertible on \( F \). Moreover, \( N \) is projective of rank equal to the degree of \( Q \) and the characteristic power series of \( \phi \) on \( N \) is \( Q \). The projectors onto the submodules \( N \) and \( F \) are in the closure of \( \mathcal{O}(Y')[\phi] \), so by the commutativity assumption these submodules are preserved under all of the endomorphisms associated to elements of \( \mathbf{T} \). Let \( \mathbf{T}(Y) \) denote the \( \mathcal{O}(Y') \)-subalgebra of \( \text{End}_{\mathcal{O}(Y')}(N) \) generated by the endomorphisms \( t_X \hat{\otimes} 1 \) for \( t \in \mathbf{T} \). This algebra is finite over \( \mathcal{O}(Y') \) and therefore affinoid.

Since the polynomial \( Q \) is the characteristic power series of \( \phi \) on \( N \) and has a unit for a leading coefficient, \( \phi \) is invertible on \( N \). Moreover, since \( Q^*(\phi) = 0 \) on \( N \), \( Q(\phi^{-1}) = 0 \) on \( N \) as well. Thus we have well-defined map

\[
\mathcal{O}(Y) \cong \mathcal{O}(Y')[T]/(Q(T)) \rightarrow \mathbf{T}(Y)
\]

\[
T \mapsto \phi^{-1}
\]

which is to say that the affinoid \( D_Y = \text{Sp}(\mathbf{T}(Y)) \) is equipped with a natural finite map \( D_Y \rightarrow Y \). These affinoids and maps can be glued together for varying \( Y \in \mathcal{C}(Z_X) \) to obtain a space \( D_X \) equipped with a map \( D_X \rightarrow Z_X \). Finally, using the links \( \alpha_{ij} \) we can glue over varying \( X \subseteq W \) to obtain a space \( D \) and canonical maps

\[
D \rightarrow Z \rightarrow W.
\]
These spaces and maps have a particularly nice interpretation on the level of points. Let $L$ be a complete and discretely-valued extension of $K$.

**Definition 3.2.** — A pair $(\kappa, \gamma)$ consisting of an $L$-valued point $\kappa \in W(L)$ and a map of sets $\gamma : T \rightarrow L$ is an $L$-valued system of eigenvalues of $T$ acting on the $\{ M_X \}$ if there exists an admissible open affinoid $X \subseteq W$ containing $\kappa$ and a nonzero element

$$m \in M_X \hat{\otimes}_{O(X), \kappa L}$$

such that

$$(t \otimes 1)m = \gamma(t)m$$

for all $t \in T$. This system of eigenvalues is called $\phi$-finite if $\gamma(\phi) \neq 0$.

Let $x$ be an $L$-valued point of $D$. Then $x$ lives over a point $\kappa_x$ in some admissible affinoid open $X \subseteq W$ and moreover lies in $DY$ for some $Y \in \mathcal{C}(Z_X)$. The associated $K$-algebra map $T(Y) \rightarrow L$ gives a map $\gamma_x : T \rightarrow L$ of sets.

**Lemma 3.3.** — The association $x \mapsto (\kappa_x, \gamma_x)$ is a well-defined bijection between the set of $L$-valued points of $D$ and the set of $L$-valued $\phi$-finite systems of eigenvalues of $T$ acting on the $\{ M_X \}$. The map $D \rightarrow Z$ is given by

$$x \mapsto (\kappa_x, \gamma_x(\phi)^{-1})$$

on $L$-valued points.

**Proof.** — The first assertion is proven in [1]. The second is obvious from the definition of the map $D \rightarrow Z$. 

For the remainder of the paper, $p$ will denote an odd prime and $W$ will denote $p$-adic weight space over $\mathbb{Q}_p$. Fix a positive integer $N$ prime to $p$. For each admissible affinoid open $X \subseteq W$ and each rational number $r \in [0, r_n]$, we define $M_X(N, \mathbb{Q}_p, p^{-r})$ to be the $\mathcal{O}(X)$-Banach module of families of (integral weight) modular forms of tame level $N$ and growth condition $p^{-r}$ on $X$. This module has been defined, for example, in [4] and [1] where an action of the Hecke operators $T_{\ell}$ ($\ell \nmid Np$) and $U_{\ell}$ ($\ell \mid Np$) and the tame diamond operators $\langle d \rangle_N$ ($d \in (\mathbb{Z}/N\mathbb{Z})^\times$) is also defined. These operators are continuous and the operator $U_{\ell}$ is compact whenever $r > 0$. Similarly, we define $\tilde{M}_X(4N, \mathbb{Q}_p, p^{-r})$ to be the $\mathcal{O}(X)$-Banach module of families of half-integral weight modular forms of tame level $4N$ and growth condition $p^{-r}$ on $X$. This module was introduced in [9] where an action of the Hecke operators $T_{p^2}$ ($\ell \nmid 4Np$) and $U_{p^2}$ ($\ell \mid 4Np$) and the tame diamond operators $\langle d \rangle_{4N}$ ($d \in (\mathbb{Z}/4N\mathbb{Z})^\times$) is also defined. These operators are continuous and $U_{p^2}$ is compact whenever $r > 0$. Each of these modules has a cuspidal submodule having all of the same operators and properties and will be denoted by replacing the letter $M$ by the letter $S$. The tilde will be used throughout the paper to distinguish half-integral weight objects from their integral weight counterparts.

All of these modules of forms satisfy $(Pr)$ of [1]. The system of Banach modules $\{ S_X \}$, where $S_X = S_X(N, \mathbb{Q}_p, p^{-r_n})$ and $n$ is chosen to be the smallest integer such that $X \subseteq W_n$, carries canonical links defined in [1] (strictly speaking on the entire space of forms, but they can be defined in exactly the same manner on the cuspidal
submodule). In [9] we construct canonical links for the system \( \{ \tilde{S}_X \} \) where \( \tilde{S}_X = \tilde{S}_X(4N, \mathbb{Q}_p, p^{-r_n}) \). The Hecke operators and tame diamond operators in each case furnish commuting endomorphisms that are compatible with these links, and \( U_p \) and \( U_{p^2} \) are compact, so we may apply the above construction in both the integral and half-integral weight cases. Note that the eigenvarieties and spectral varieties so obtained are equidimensional of dimension 1 by Lemma 5.8 of [1]. We denote the eigencurve and spectral curve associated to \( \{ S_X(N, \mathbb{Q}_p, p^{-r_n}) \} \) by \( D \) and \( Z \), respectively, and refer to them as the \textit{cuspial integral weight eigencurve} and \textit{spectral curve} of tame level \( N \), respectively. Similarly we denote the eigencurve and spectral curve associated to \( \{ \tilde{S}_X(4N, \mathbb{Q}_p, p^{-r_n}) \} \) by \( \tilde{D} \) and \( \tilde{Z} \), respectively, and refer to them as the \textit{cuspial half-integral weight eigencurve} and \textit{spectral curve} of tame level \( 4N \), respectively.

**Remark 3.4.** — In the interest of notational brevity we have chosen not to adorn these spacing so as to indicate that we are working only with the cuspidal part. This puts the notation in conflict with, for example, the author’s previous notation in [9]. Hopefully no confusion will arise from this conflict.

### 4. Some density results

We will need some lemmas on the density of certain sets of classical points in what follows. Following Chenevier ([2]), we call a subset \( \Sigma \subseteq \mathcal{W}(\mathcal{C}_p) \) \textit{very Zariski-dense} if for each \( \kappa \in \Sigma \) and each irreducible (equivalently, connected) admissible affinoid open \( V \subseteq \mathcal{W} \) containing \( \kappa \), \( V(\mathcal{C}_p) \cap \Sigma \) is Zariski-dense in \( V \).

**Lemma 4.1.** — The set \( \Sigma \) of weights of the form \( \lambda \tau^j \) for \( \lambda \geq 0 \) and \( 0 \leq j < p - 1 \) is very Zariski-dense in \( \mathcal{W} \).

**Proof.** — Fix \( \lambda_0 \tau^{j_0} \in \Sigma \) and suppose moreover that \( \lambda_0 \tau^{j_0} \in \mathcal{W}^i \). Let \( V \) be any connected admissible affinoid containing \( \lambda_0 \), so \( V \subseteq \mathcal{W}^i \). Recall that \( \mathcal{W}^i \) is isomorphic to the open unit ball \( B(1, 1) \) about 1, the isomorphism being \( \kappa \mapsto \kappa(1 + p) \).

Note that the points in \( \Sigma \) are all \( \mathbb{Q}_p \)-valued. Since \( V(\mathbb{Q}_p) \) is open in the \( p \)-adic topology, there exists an \( \epsilon > 0 \) such that the closed ball \( B(\lambda_0 \tau^{j_0}, \epsilon) \) of radius \( \epsilon \) about \( \lambda_0 \tau^{j_0} \) is contained in \( V \). Admit for the moment that \( B(\lambda_0 \tau^{j_0}, \epsilon)(\mathbb{Q}_p) \cap \Sigma \) is Zariski dense in \( B(\lambda_0 \tau^{j_0}, \epsilon) \). We claim that this implies that \( V(\mathbb{Q}_p) \cap \Sigma \) is Zariski-dense in \( V \). Let \( T \subset V \) be an analytic subset containing \( V(\mathbb{Q}_p) \cap \Sigma \). Then \( T \cap B(\lambda_0 \tau^{j_0}, \epsilon) \) is an analytic subset of \( B(\lambda_0 \tau^{j_0}, \epsilon) \) containing \( B(\lambda_0 \tau^{j_0}, \epsilon)(\mathbb{Q}_p) \cap \Sigma \) and by assumption we conclude that \( T \) contains \( B(\lambda_0 \tau^{j_0}, \epsilon) \). It now follows from Lemma 2.2.3 of [5] that \( T = V \).

So we are reduced proving Zariski-density in the special case of \( B(\lambda_0 \tau^{j_0}, \epsilon) \), which amounts to proving that this ball contains infinitely many elements of \( \Sigma \). But

\[
|\lambda \tau^j - \lambda_0 \tau^{j_0}| = |(1 + p)^\lambda - (1 + p)^{\lambda_0}| = |(1 + p)^{\lambda - \lambda_0} - 1|
\]

and we can find infinitely many \( \lambda \tau^j \in B(\lambda_0 \tau^{j_0}, \epsilon) \) which make this as small as we like by choosing \( j = j_0 \) and \( \lambda \equiv \lambda_0 \pmod{(p - 1)p^N} \) for sufficiently large \( N \). \( \square \)
Let \( X \subseteq \mathcal{W} \) be an admissible affinoid open. For any polynomial \( \tilde{h} \) over \( \mathcal{O}(X) \) in the Hecke operators \( T_{\ell^j}, U_{\ell^j} \) and the diamond operators \( \langle d \rangle_{4N} \), we denote by \( \tilde{Z}_X^k \) the subspace of \( X \times \mathbb{A}^1 \) defined as the zero locus of
\[
\tilde{P}_X^k(T) = \det(1 - \tilde{h}U_{\ell^2}T \mid \tilde{S}_X)
\]
in \( X \times \mathbb{A}^1 \).

**Definition 4.2.** Let \( K/\mathbb{Q}_p \) be a finite extension.
- We call a \( K \)-valued point \((\kappa, \alpha) \in \tilde{Z}_X^k \) classical if \( \kappa \in \Sigma \) and there exists a nonzero classical form \( F \in \tilde{S}_X^k(4N, K) \) such that \( \tilde{h}U_{\ell^2}F = \alpha^{-1}F \).
- Let \( x \in \tilde{D}(K) \) and let \((\kappa, \gamma) \) be the \( K \)-valued system of eigenvalues corresponding to \( x \) by Lemma 3.3. Then \( x \) is called classical if \( \kappa \in \Sigma \) and there exists a nonzero classical form \( F \in \tilde{S}_X^k(4N, K) \) on which the the operators \( T_{\ell^2}, U_{\ell^2}, \) and \( \langle d \rangle_{4N} \) act through \( \gamma \).

**Lemma 4.3.** Let \( X \subseteq \mathcal{W} \) be a connected admissible open affinoid containing an element of \( \Sigma \). Then the classical points are Zariski-dense in \( \tilde{Z}_X^k \).

**Proof.** Let \( C \) be an irreducible component of \( \tilde{Z}_X^k \). By Theorem 4.2.2 of [5], \( C \) is a Fredholm hypersurface and hence has Zariski-open in image in \( X \). It follows that \( C \) contains a point \( x \) mapping to an element of \( \Sigma \) by Lemma 4.1. Let \( Y \) be an element of the canonical cover \( \mathcal{C}(\tilde{Z}_X^k) \) containing \( x \) with connected image \( Y' \subseteq X \). By Lemma 4.1, \( \Sigma \cap Y' \) is Zariski-dense in \( Y' \). Moreover, the proof of the lemma shows that this is also true after omitting any finite collection of points in \( \Sigma \).

Let \( \Sigma' \) denote the direct summand of \( \tilde{S}_X \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(Y') \) corresponding to the choice of \( Y' \). The \( \mathcal{O}(Y') \)-module \( \Sigma' \) is projective of rank equal to the degree of \( Y \rightarrow Y' \) and is stable under the endomorphism \( U_{\ell^2} \). Moreover \( U_{\ell^2} \) acts invertibly on \( \Sigma' \) since \( \tilde{h}U_{\ell^2} \) does. It follows that the eigenvalues of \( U_{\ell^2} \) are bounded away from 0 on \( \Sigma' \). That is, there exists a positive integer \( M \) such that for all finite extensions \( K/\mathbb{Q}_p \) and all \( K \)-valued points \( \mathcal{O}(Y') \rightarrow K \), the roots of the characteristic polynomial of \( U_{\ell^2} \) acting on the fiber of \( N \otimes_{\mathcal{O}(Y')} K \) have absolute value at least \( p^{-M} \). This can be seen, for example, by examining the variation of the Newton polygon of the characteristic polynomial
\[
R(T) = \det(T - U_{\ell^2} \mid N)
\]
over \( Y' \). Let \( \Sigma' \) denote the complement of the finite collection of \( \lambda \tau^j \in \Sigma \) with \( 2\lambda - 1 \leq M \), and note that \( \Sigma' \cap Y' \) is still Zariski dense in \( Y' \) by the above comments. Since \( Y \rightarrow Y' \) is finite and flat, each irreducible component of \( Y \) surjects onto \( Y' \), and it follows that the preimage of \( \Sigma' \) in \( Y \) is Zariski-dense in every component of \( Y \). If \((\lambda \tau^j, \alpha) \) is a \( K \)-valued point in this preimage, then there is a nonzero overconvergent form
\[
F \in N \otimes_{\mathcal{O}(Y')} K \subseteq \tilde{S}^k_{\lambda \tau^j}(4N, K)
\]
with \( \tilde{h}U_{\ell^2}F = \alpha^{-1}F \). Since \( F \) is annihilated by \( R_{\lambda \tau^j}(U_{\ell^2}) \), the characteristic polynomial (2) with coefficients evaluated at \( \lambda \tau^j \), it follows from Theorem 2.4 that \( F \) is a
classical cusp form. Thus the classical locus is also dense in each component of \( Y \). By Corollary 2.2.9 of \([5]\), \( Y \cap C \) is a nonempty (since it contains \( x \)) union of irreducible components of \( Y \). It follows easily from Lemma 2.2.3 of \([5]\) that a Zariski-dense subset of an admissible open in an irreducible space is in fact Zariski dense in the whole space, so we conclude that the classical points are Zariski-dense in \( C \) for each \( C \), and the lemma follows.

\[ \text{Remark 4.4.} \quad \text{In case } \tilde{h} = 1, \text{ this proof also shows that the set of points in } \tilde{Z}_X \text{ of the form } (\lambda \tau^j, \alpha) \text{ with } v(\alpha^{-1}) < 2\lambda - 1 \text{ is Zariski-dense in } \tilde{Z}_X. \text{ Since } W \text{ is admissibly covered by a collection of connected admissible affinoid subdomains } \{X\} \text{ each of which meets } \Sigma (\text{such as } \{W^n_i\}) \text{ we conclude that the set of } (\lambda \tau^j, \alpha) \in \tilde{Z} \text{ with } v(\alpha^{-1}) < 2\lambda - 1 \text{ is Zariski-dense in all of } \tilde{Z}. \]

\[ \text{Corollary 4.5.} \quad \text{The classical points are Zariski-dense in } \tilde{D}. \]

\[ \text{Proof.} \quad \text{By Lemma 5.8 of } [1], \text{ the finite map } \tilde{D} \rightarrow \tilde{Z} \text{ carries every irreducible component of } \tilde{D} \text{ surjectively onto an irreducible component of } \tilde{Z}. \text{ It follows from this and Remark 4.4 that the set of points } x \in \tilde{D} \text{ such that the corresponding system of eigenvalues } (\kappa, \alpha) \text{ has } \kappa \in \Sigma \text{ and } v(\alpha^{-1}) < 2\lambda - 1 \text{ is Zariski-dense in } \tilde{D}. \text{ But by Theorem 2.4, all such points are in fact classical.} \]

5. Interpolation of the Shimura lifting

Let \( N \) be a positive integer not divisible by the odd prime \( p \). For the remainder of the paper all spaces of modular forms of half-integral weight will be taken at tame level \( 4N \) and all spaces of modular forms of integral weight will be taken at tame level \( 2N \). Let

\[ 2 : W \rightarrow W \]

denote the finite map given by \( 2(\kappa) = \kappa^2 \) on the level of points. We wish to construct maps \( \text{Sh} : \tilde{D}_{\text{red}} \rightarrow D_{\text{red}} \) and \( \tilde{Z}_{\text{red}} \rightarrow Z_{\text{red}} \) fitting into the diagram

\[ (3) \]

\[ \begin{array}{ccc}
\tilde{D}_{\text{red}} & \xrightarrow{\text{Sh}} & D_{\text{red}} \\
\downarrow & & \downarrow \\
\tilde{Z}_{\text{red}} & \rightarrow & Z_{\text{red}} \\
\downarrow & & \downarrow \\
W & \xrightarrow{2} & W \\
\end{array} \]

whose vertical arrows are the canonical ones arising from the constructions of the eigencurves and such that if \( x \) is a classical point then \( \text{Sh}(x) \) is the system of eigenvalues of the classical Shimura lift of a classical form corresponding to \( x \). The map \( \tilde{Z}_{\text{red}} \rightarrow Z_{\text{red}} \) will simply be given on points by

\[ (\kappa, \alpha) \mapsto (\kappa^2, \alpha), \]

though it is not yet at all clear that this map is well-defined.
We record a couple of lemmas concerning nilreductions of Fredholm varieties that we will need in the sequel.

**Lemma 5.1.** — With notation as in Theorem 3.1, assume moreover that $R$ is relatively factorial. Then the map

$$f: Z \hookrightarrow Z_{\text{red}}$$

is a bijection.

**Proof.** — First note that $Z_{\text{red}}$ is Fredholm by Theorem 4.2.2 of [5]. Let $Y \in C(Z)$ with image $Y' \subseteq \text{Sp}(R)$. That $Y \hookrightarrow Y'$ is finite implies that $Y_{\text{red}} \hookrightarrow Y'_{\text{red}} = Y'$ is finite. To get an idempotent that cuts out $Y_{\text{red}}$, simply pull back the idempotent that cuts out $Y$ through the canonical reduction map, so the proposed map at least makes sense.

Let $X \in C(Z_{\text{red}})$ with image $Y' \subseteq \text{Sp}(R)$. By the proof of A1.1 in [6], the map $Z_{\text{red}} \hookrightarrow Z$ is a homeomorphism of Grothendieck topologies. In particular the underlying open set of $X$ is also an admissible open in $Z$. As such, it inherits the structure of a rigid space by restricting the structure sheaf of $Z$ to $X$. Let $Y$ denote the rigid space so obtained. I claim that $Y \in C(Z)$ and that the map $X \mapsto Y$ is the inverse to the above map.

Since reduction and passing to an admissible open commute, $Y_{\text{red}} = X$. Now Theorem A1.1 of [6] implies that $Y \mapsto Y'$ is finite, so the comments following Theorem 3.1 imply that $Y \in C(Z)$. That these two maps are inverse to each other is clear. □

If $F$ is a Fredholm series over a relatively factorial affinoid, let $F_{\text{red}}$ denote the unique Fredholm series such that $Z(F_{\text{red}}) = Z(F)_{\text{red}}$ (the existence and uniqueness of such a series is guaranteed by Theorem 4.2.2 of [5]).

**Lemma 5.2.** — Let $A$ and $B$ be relatively factorial affinoid algebras over $K$ with $A$ an integral domain and let $f: \text{Sp}(A) \hookrightarrow \text{Sp}(B)$ be a map of rigid spaces over $K$. Let $F$ and $G$ be Fredholm series over $A$ and $B$ respectively. The following are equivalent.

(a) $F_{\text{red}}$ divides $(f^*G)_{\text{red}}$ in the ring of entire series over $A$.

(b) $f^*G$ vanishes on the zero locus of $F$.

(c) $Z(F)_{\text{red}}$ is a union of irreducible components of

$$Z(G) \times_{\text{Sp}(B)} \text{Sp}(A)_{\text{red}} = Z(f^*G)_{\text{red}}.$$ 

(d) there exists a unique map

$$Z(F)_{\text{red}} \mapsto Z(G)_{\text{red}}$$

that is given by $(x, \alpha) \mapsto (f(x), \alpha)$ on points.

**Proof.** — Since passing to the reduction does not change the zero locus, (a) implies (b) trivially. Suppose that (b) holds. Then $(f^*G)_{\text{red}}$ also vanishes on $Z(F)$, and (c) follows from Lemma 4.1.1 of [5]. Now suppose that (c) holds. By Corollary 2.2.6 of [5], $Z(F)_{\text{red}}$ is a “component part” of $Z(f^*G)_{\text{red}}$ in the sense of [4]. By Proposition
1.3.4 of \cite{4}, \(F_{\text{red}}\) divides \((f^*G)_{\text{red}}\), and we have (a). It remains to see that (d) is equivalent to these first three conditions. Suppose that that (c) holds. The composite map 

\[ Z(F)_{\text{red}} \hookrightarrow Z(f^*G)_{\text{red}} \twoheadrightarrow Z(f^*G) = Z(G) \times \text{Sp}(B) \text{Sp}(A) \twoheadrightarrow Z(G) \]

factors through a unique map \(Z(F)_{\text{red}} \twoheadrightarrow Z(G)_{\text{red}}\) by the universal property of reduction. This map has the desired effect on points and is the unique one with this property since these spaces are reduced. Finally, that (d) implies (b) is clear. \(\square\)

**Proposition 5.3.** — Let \(X\) and \(\bar{X}\) be connected admissible affinoid subdomains of \(\mathcal{W}\) such that \(2\) restricts to a map \(2 : \bar{X} \twoheadrightarrow X\), and suppose that \(\bar{X}\) contains a point of \(\Sigma\). Let \(h\) be a polynomial over \(\mathcal{O}(X)\) in the symbols \(T_{\ell}, U_{\ell}\), and \((d)_{2N}\) and let \(\tilde{h}\) denote the corresponding polynomial obtained by replacing these symbols by \(T_{\ell^2}, U_{\ell^2}\), and \((d^2)_{4N}\), respectively, and pulling back the coefficients to \(\mathcal{O}(\bar{X})\). Let 

\[ P^h_X(T) = \det(1 - hU_pT \mid S_X) \]

and 

\[ \tilde{P}^h_{\bar{X}}(T) = \det(1 - \tilde{h}U_p^2T \mid \tilde{S}_{\bar{X}}). \]

Then \(\tilde{P}^h_{\bar{X}}(T)_{\text{red}}\) divides \(2^*P^h_X(T)_{\text{red}}\).

**Proof.** — By Lemma 5.2 it suffices to check that \(2^*P^h_X(T)\) vanishes on the zero locus of \(P^h_{\bar{X}}\). By Lemma 4.3, the classical points are dense in this locus. The same is true of the set of classical points of weight \(\lambda \tau^j\) with \(\lambda \geq 2\) since this omits only finitely many points in \(\Sigma\). Let \((\lambda \tau^j, \alpha)\) be a point in the zero locus of \(P^h_{\bar{X}}\) with \(\lambda \geq 2\) and \(\alpha\) in a finite extension \(K\) of \(\mathbb{Q}_{p}\) and define \(k = 2\lambda + 1\). Then the space 

\[ V = \{ F \in \tilde{S}_{k/2}'(4N, K)^{\tau_j} \mid \tilde{h}_{\lambda \tau^j}U_p^2F = \alpha^{-1}F \}, \]

where \(\tilde{h}_{\lambda \tau^j}\) denotes the polynomial \(\tilde{h}\) with coefficients evaluated at \(\lambda \tau^j\), is nonzero. Since all of the operators \(T_{\ell^2}, U_{\ell^2}\), and \((d)_{4N}\) commute with \(\tilde{h}_{\lambda \tau^j}U_p^{2}\), they act on the space \(V\) and hence there exists a finite extension \(L/K\) and a nonzero element \(F \in \tilde{S}_{k/2}'(4N, L)^{\tau_j}\) that is a simultaneous eigenform for all of these operators. In particular it has a character \(\chi\) for the action of the \((d)_{4N}\), and by Theorem 2.5 there exists a nonzero classical form \(f\) of level 

\(2N\) and weight \(k - 1\) defined over \(K\) that lies in the \(\tau^j\)-eigenspace for \((d)_{p}\) such that 

\[ h_{(2\lambda)^{\tau^j}}U_p^j f = \alpha^{-1}f. \]

Since \(\alpha, \tau(d) \in K\) for all \(d \in (\mathbb{Z}/p\mathbb{Z})^\times\), there must also be a form \(f\) defined over \(K\) with these properties. Thus \((2\lambda)^{\tau^j, \alpha}\) is a root of \(P^h_X\), which is to say that \((\lambda \tau^j, \alpha)\) is a root of \(2^*P^h_X\). \(\square\)

Let \(X \subseteq \mathcal{W}\) be an admissible affinoid open and let \(\kappa \in \mathcal{W}(K)\) be a point. For any module \(M\) over \(\mathcal{O}(X)\) we denote by \(M_\kappa\) the vector space \(M \otimes_{\mathcal{O}(X)} K\) and for any power series \(P\) over \(\mathcal{O}(X)\) we denote by \(P_\kappa\) the power series over \(K\) obtained by evaluating the coefficients of \(P\) at \(\kappa\).
Corollary 5.4. — Let $\kappa \in \mathcal{W}(K)$, let $h_0$ be a polynomial over $K$ in the symbols $T_\ell$, $U_\ell$, and $(d)_{2N}$, and let $\tilde{h}_0$ denote the polynomial obtained by replacing these symbols by $T_\ell$, $U_\ell$, and $(d^2)_{4N}$, respectively. Pick a connected admissible affinoid open $\tilde{X}$ containing $\kappa$ and let $X = 2(\tilde{X})$. Then

$$\det(1 - \tilde{h}_0U_\ell T | (\tilde{S}_\kappa)_{\kappa})_{\text{red}} \mid \det(1 - h_0U_\ell T | (S_X)_{\kappa^2})_{\text{red}}$$

Proof. — Note that $X$ is necessarily a connected admissible affinoid (and moreover $2 : \tilde{X} \rightarrow X$ is an isomorphism), so the assertion at least makes sense. By enlarging $\tilde{X}$ if necessary we may assume that $\tilde{X}$ contains an element of $\Sigma$, since the links $\alpha_{ij}$ ensure that this enlargement does not affect the claimed divisibility. Let $h$ be any polynomial over $O(X)$ in the symbols $T_\ell$, $U_\ell$, and $(d)_{2N}$ with $h_{\kappa^2} = h_0$, and let $\tilde{h}$ denote the polynomial over $O(\tilde{X})$ obtained by replacing these symbols by $T_\ell$, $U_\ell$, and $(d^2)_{4N}$, respectively, and pulling back the coefficients via $2^*$. Clearly we have $\tilde{h}_\kappa = \tilde{h}_0$. By Proposition 5.3 we have

$$\det(1 - \tilde{h}_0U_\ell T | (\tilde{S}_\kappa)_{\kappa})_{\text{red}} \mid (2^* \det(1 - hU_\ell T | S_X))_{\text{red}}.$$  

The result now follows from Lemma 2.13 of [1] and Lemma 5.2 by specializing to $\kappa$.

Corollary 5.5. — Let $\tilde{X} \subseteq \mathcal{W}$ be a connected admissible affinoid open and let $X = 2(\tilde{X})$. There is a unique finite map $\tilde{Z}_{\tilde{X}, \text{red}} \rightarrow Z_{X, \text{red}}$ having the effect $(\kappa, \alpha) \mapsto (\kappa^2; \alpha)$ on points.

Proof. — Choose integers $i$ and $n$ with $\tilde{X} \subseteq \mathcal{W}_n^i$, so that $X \subseteq \mathcal{W}_n^{2i}$. By Proposition 5.3 (with $\tilde{h} = 1$) and Lemma 5.2, $2^*P_{\mathcal{W}_n^i}(T)$ vanishes on the zero locus of $\tilde{P}_{\mathcal{W}_n^2}(T)$. Let $i$ and $\tilde{i}$ denote the inclusions of $X$ and $\tilde{X}$ into $\mathcal{W}_n^{2i}$ and $\mathcal{W}_n^i$, respectively. Then

$$\tilde{i}^*2^*P_{\mathcal{W}_n^i}(T) = 2^*\tilde{i}^*P_{\mathcal{W}_n^2}(T)$$

vanishes on the zero locus of $\tilde{i}^*P_{\mathcal{W}_n^i}(T)$. Lemma 2.13 and the links $\alpha_{ij}$ ensure that

$$\tilde{i}^*P_{\mathcal{W}_n^i}(T) = P_X(T) \quad \text{and} \quad \tilde{i}^*\tilde{P}_{\mathcal{W}_n^i}(T) = \tilde{P}_X(T).$$

The existence of a map having the indicated effect on points now follows from Lemma 5.2, and it remains to see that this map is finite. But by Lemma 5.2 the map is the composition of the inclusion of the union of irreducible components $\tilde{Z}_{\tilde{X}, \text{red}}$ into $(Z_X \times_X \tilde{X})_{\text{red}}$ and the (nilreduction of) the projection $Z_X \times_X \tilde{X} \rightarrow Z_X$. The former is obviously finite and the latter is finite because the map $2 : \tilde{X} \rightarrow X$ is an isomorphism.

For each $i$, the $\mathcal{W}_n^i$ form a nested sequence, so we may glue the maps furnished by Corollary 5.5 applied to $\tilde{X} = \mathcal{W}_n^i$ and $X = \mathcal{W}_n^{2i}$ over increasing $n$ to obtain a diagram

$$\begin{array}{ccc}
\tilde{Z}_{\mathcal{W}_n^i, \text{red}} & \longrightarrow & Z_{\mathcal{W}_n^{2i}, \text{red}} \\
\downarrow & & \downarrow \\
\mathcal{W}_n^i & \overset{2}{\longrightarrow} & \mathcal{W}_n^{2i}
\end{array}$$
for each $0 \leq i < p - 1$. Now since $\mathcal{W}$ is the disjoint union of the $\mathcal{W}^i$, we obtain the bottom square in the diagram (3). Let $g : \tilde{Z}_{\text{red}} \rightarrow Z_{\text{red}}$ be the map so obtained. On the level of points, $g$ is simply given by $g(\kappa, \alpha) = (\kappa^2, \alpha)$, as desired. The next lemma shows that $g$ interacts well with the canonical covers of its source and target.

**Lemma 5.6.** — Let $\tilde{X} \subseteq \mathcal{W}$ be a connected admissible affinoid open and let $X = 2(\tilde{X})$. Let $Y$ be an element of the canonical cover $\mathcal{C}(Z_X)$ of $Z_X$ with connected image $Y' \subseteq X$. Then $g^{-1}(Y_{\text{red}})$ is either empty or is an element of $\mathcal{C}(\tilde{Z}_{\tilde{X}, \text{red}})$ with connected image $2^{-1}(Y')$.

**Proof.** — Since $g$ and 2 are finite, $g^{-1}(Y_{\text{red}})$ and $2^{-1}(Y')$ are affinoids. The latter is connected since $Y'$ is connected and $2 : \tilde{X} \rightarrow X$ is an isomorphism.

By the construction of $g$ (see the proof of Lemma 5.2), $g^{-1}(Y_{\text{red}})$ is the intersection inside $Z_{X, \text{red}} \times_X \tilde{X}$ of the admissible affinoid $Y_{\text{red}} \times_X \tilde{X}$ and the union of irreducible components $\tilde{Z}_{\tilde{X}, \text{red}}$. This intersection is a (possibly empty) union of irreducible components of the admissible affinoid $Y_{\text{red}} \times_X \tilde{X}$ by Corollary 2.2.9 of [5]. But

$$Y_{\text{red}} \times_X \tilde{X} \cong Y_{\text{red}} \times Y' \cdot 2^{-1}(Y')$$

is finite over $2^{-1}(Y')$ since $Y_{\text{red}}$ is finite over $Y'$, and hence so is any nonempty subspace of components, such as $g^{-1}(Y_{\text{red}})$ (if nonempty).

That $g^{-1}(Y_{\text{red}})$ is disconnected from its complement in the full preimage of $2^{-1}(Y')$ follows from the analogous property of $Y_{\text{red}}$; one simply pulls back the idempotent that cuts out $Y_{\text{red}}$ though $g$ to get one that cuts out $g^{-1}(Y_{\text{red}})$.

Let $X, \tilde{X},$ and $Y$ be as in the previous lemma and assume that $g^{-1}(Y) \neq \emptyset$ so that $g^{-1}(Y_{\text{red}})$ is in $\mathcal{C}(\tilde{Z}_{\tilde{X}, \text{red}})$ with connected image $2^{-1}(Y')$. Let

$$P_{Y'}(T) = Q(T)Q'(T)$$

and

$$\tilde{P}_{2^{-1}(Y')}(T) = \tilde{Q}(T)\tilde{Q}'(T)$$

denote the factorizations arising from the choice of $Y$ and the $\tilde{Y} \in \mathcal{C}(\tilde{X})$ of which $g^{-1}(Y_{\text{red}})$ is the underlying reduced affinoid (this well-defined by Lemma 5.1). Since $g$ restricts to maps

$$g^{-1}(Y_{\text{red}}) \rightarrow Y_{\text{red}} \quad \text{and} \quad \tilde{Z}_{2^{-1}(Y'), \text{red}} \setminus g^{-1}(Y_{\text{red}}) \rightarrow Z_{Y', \text{red}} \setminus Y_{\text{red}},$$

Lemma 5.2 ensures that $\tilde{Q}_{\text{red}}|_{(2^iQ)_{\text{red}}}$ and $\tilde{Q}'_{\text{red}}|_{(2^iQ')_{\text{red}}}$. Let $A$ and $B$ denote the affinoid algebras of $Y'$ and $2^{-1}(S)$, respectively. Let

$$S_X \hat{\otimes}_{O(X)} A = N \oplus F$$

and

$$\tilde{S}_{O(\tilde{X})} \hat{\otimes} B \cong \tilde{N} \oplus \tilde{F}$$

denote the corresponding decompositions of the spaces of families of cusp forms.

To construct the $p$-adic Shimura lift $\mathcal{L}$ and complete the diagram (3), we will construct the part covering $g : g^{-1}(Y_{\text{red}}) \rightarrow Y_{\text{red}}$ for each $Y$ and glue. Let $T(Y)$ denote the $A$-subalgebra of $\text{End}_R(N)$ generated by $T_\ell, U_\ell,$ and $\langle d \rangle_{2N}$ and let $\tilde{T}(Y)$
denote the $B$-subalgebra of $\text{End}_S(\tilde{N})$ generated by $T_\ell$, $U_\ell$, and $\langle d \rangle_{4N}$. We wish to show that the map

\begin{align*}
T(Y)_{\text{red}} &\quad \mapsto \quad \tilde{T}(Y)_{\text{red}} \\
T_\ell &\quad \mapsto \quad T_\ell \\
U_\ell &\quad \mapsto \quad U_\ell \\
\langle d \rangle_{2N} &\quad \mapsto \quad \langle d^2 \rangle_{4N}
\end{align*}

that is given by $2^*: A \mapsto B$ on coefficients, is well-defined. The following lemma furnishes the key divisibility needed to prove this well-definedness.

**Lemma 5.7.** Let $h$ be a polynomial over $A$ in the symbols $T_\ell$, $U_\ell$, and $\langle d \rangle_{2N}$, and let $\tilde{h}$ be the polynomial over $B$ obtained by replacing these symbols by $T_\ell$, $U_\ell$, and $\langle d^2 \rangle_{4N}$, respectively, and applying the map $2^*: A \mapsto B$ to the coefficients. Then

$$\det(1 - \tilde{h}U_\ell^2T | \tilde{N})_{\text{red}} \mid \langle 2^* \det(1 - hU_\ell^2T | N) \rangle_{\text{red}}.$$ 

**Proof.** By Lemma 5.2, the divisibility claimed in the statement of the lemma can be checked after specializing to each $\kappa \in 2^{-1}(Y')$. Define

$$\tilde{S}_\kappa = \tilde{S}_X \hat{\otimes} \mathcal{O}(\tilde{X}),_{\kappa} K \quad \text{and} \quad S_{\kappa^2} = S_X \hat{\otimes} \mathcal{O}(X),_{\kappa^2} K$$

and define $\tilde{N}_\kappa$ and $N_{\kappa^2}$ similarly. By Lemma 2.13 of [1], this amounts to checking that

\begin{equation}
\det(1 - \tilde{h}_\kappa U_\ell^2T | \tilde{N}_\kappa)_{\text{red}} \mid \det(1 - h_{\kappa^2} U_\ell^2T | N_{\kappa^2})_{\text{red}}
\end{equation}

for each $\kappa \in 2^{-1}(Y')$.

For any $\sigma \in \mathbb{R}$, we define $\tilde{S}_\kappa^\sigma$ ($S_{\kappa^2}^\sigma$, $\ldots$, etc.) to be the slope $\sigma$ subspace for the relevant operator ($U_p$ for integral weight spaces and $U_{p^2}$ for half-integral weight spaces). These spaces are all finite-dimensional and $\tilde{N}_\kappa$ is moreover the direct sum of the subspaces $\tilde{N}_\kappa^\sigma$ that are nonzero since $U_{p^2}$ is invertible on $\tilde{N}_\kappa$, and similarly for $N_{\kappa^2}$. Let $h_0$ be any polynomial over $K$ in the symbols $T_\ell$, $U_\ell$, and $\langle d \rangle_{2N}$ and let $\tilde{h}_0$ be the polynomial obtained by replacing the symbols by $T_\ell$, $U_\ell$, and $\langle d^2 \rangle_{4N}$, respectively (no need to pull back the coefficients). We claim that

\begin{equation}
\det(T - \tilde{h}_0 | \tilde{S}_\kappa^\sigma)_{\text{red}} \mid \det(T - h_0 | S_{\kappa^2}^\sigma)_{\text{red}}
\end{equation}

for all $\sigma$. Since the endomorphisms of $N$ and $\tilde{N}$ associated to $h_0$ and $\tilde{h}_0$, respectively, are continuous, their eigenvalues are bounded and we can find a single nonzero $x \in K^\times$ such that both $h_0' = 1 + x h_0$ and $\tilde{h}_0' = 1 + x \tilde{h}_0$ have all eigenvalues of absolute value 1. It follows that

$$\det(1 - \tilde{h}_0^\sigma U_\ell^2T | \tilde{S}_\kappa^\sigma) = \det(1 - h_0^\sigma U_\ell^2T | S_{\kappa^2}^\sigma)$$

and

$$\det(1 - h_0^\sigma U_\ell^2T | S_{\kappa^2}^\sigma) = \det(1 - \tilde{h}_0^\sigma U_\ell^2T | S_{\kappa^2}^\sigma)$$

where by $F(T)^\sigma$ for a Fredholm determinant $F(T)$ over $K$ we mean the unique (polynomial) Fredholm factor with the property that the Newton polygon of $F(T)^\sigma$ has
pure slope $\sigma$ and the Newton polygon of $F(T)/F(T)^{\sigma}$ has no sides of slope $\sigma$. But now it follows from Corollary 5.4 that
\[
\det(1 - h_0' U_p T | \tilde{S}_\kappa^\sigma)_{\text{red}} \mid \det(1 - h_0 U_p T | S_\kappa^\sigma)_{\text{red}}
\]
and since $\tilde{h}_0 U_p$ and $h_0' U_p$ are invertible on these spaces, we can deduce the analogous divisibility for characteristic polynomials,
\[
\det(T - \tilde{h}_0 U_p | \tilde{S}_\kappa^\sigma)_{\text{red}} \mid \det(T - h_0' U_p | S_\kappa^\sigma)_{\text{red}}
\]
as well. By the same reasoning, this divisibility holds after replacing $h_0'$ and $\tilde{h}_0'$ by $h_0' + \epsilon$ and $\tilde{h}_0' + \epsilon$ all sufficiently small $\epsilon \in K$. Let
\[
\det(T - (h_0' + X) U_p | \tilde{S}_\kappa^\sigma) = \prod_i (T - \tilde{a}_i X - \tilde{b}_i)
\]
and
\[
\det(T - (h_0' + X) U_p | S_\kappa^\sigma) = \prod_j (T - a_j X - b_j)
\]
Then we have shown that for infinitely many $\epsilon \in K$ it is the case that for each $i$ there exists $j$ such that
\[
\tilde{a}_i \epsilon + \tilde{b}_i = a_j \epsilon + b_j.
\]
It follows easily that for each $i$ there exists $j$ such that $\tilde{a}_i = a_j$ and $\tilde{b}_i = b_j$. Simultaneously upper-triangularizing the commuting endomorphisms $\tilde{h}_0'$ and $U_p$, we see that the $\tilde{b}_i/\tilde{a}_i$ are exactly the eigenvalues of $h_0'$ on $\tilde{S}_\kappa^\sigma$. Similarly, the $b_j/a_j$ are the eigenvalues of $h_0'$ on $S_\kappa^\sigma$, and we conclude that
\[
\det(T - \tilde{h}_0' | \tilde{S}_\kappa^\sigma)_{\text{red}} \mid \det(T - h_0' | S_\kappa^\sigma)_{\text{red}}.
\]
Now (6) follows from a linear change of variables.

We claim that
\[
\det(1 - \tilde{h}_0 U_p X | \tilde{N}_\kappa^\sigma)_{\text{red}} \mid \det(1 - h_0 U_p T | N_\kappa^{\sigma})_{\text{red}}
\]
for each $\sigma \in \mathbb{R}$. In particular, this establishes the desired divisibility (5) by the comments that follow it. Let $\alpha$ be a root of
\[
\det(1 - \tilde{h}_0 U_p X | \tilde{N}_\kappa^\sigma).
\]
By enlarging $K$ if necessary (the only requirement on $K$ in the preceding argument is that it be finite over $\mathbb{Q}_p$ and contain the residue field of $\kappa$) we may assume that $\alpha \in K$. Let $\mathcal{H}$ denote the free commutative $K$-algebra generated by the symbols $T_\ell$, $U_\ell$, and $\langle d \rangle_{2N}$. The finite-dimensional $K$-vector space $S_\kappa^\sigma \oplus \tilde{S}_\kappa^\sigma$ is a finite length algebra over $\mathcal{H}$, where $\mathcal{H}$ acts in the obvious way on $S_\kappa^\sigma$ and on $\tilde{S}_\kappa^\sigma$ we agree that $T_\ell$ acts by $T_\ell$, $U_\ell$ acts by $U_\ell$, and $\langle d \rangle_{2N}$ acts by $\langle d^2 \rangle_{4N}$. Let $\tilde{W}$ be a simple (over $\mathcal{H}$) constituent of
\[
\{ F \in \tilde{N}_\kappa^\sigma | \tilde{h}_0 U_p F = \alpha^{-1} F \}_{\text{ms}} \neq 0,
\]
where for a (finite length) $\mathcal{H}$-module $\mathcal{M}$, $\mathcal{M}^{\text{ms}}$ denotes the semisimplification of $\mathcal{M}$ as an $\mathcal{H}$-module. By general facts about semisimple algebras, there exists $h_0 \in \mathcal{H}$ such that $h_0$ acts via the identity on $\tilde{W}$ and $h_0 = 0$ on any simple constituent of
Problem 5.8. — Let $N$ be a positive integer and let $p$ be an odd prime not dividing $N$. Let $D$ and $Z$ denote the integral weight cuspidal eigencurve and spectral curve of level $2N$, respectively, and let $\tilde{D}$ and $\tilde{Z}$ denote the half-integral weight cuspidal

$S_{\kappa^2}^\text{ss} \oplus \bar{S}_{\kappa}^\text{ss}$ that is not isomorphic to $W$. Divisibility (6) implies that 1 is a root of $\det(T - h_{0} \mid S_{\kappa^2}^\text{ss})$, so there must be a simple constituent $W$ of $S_{\kappa^2}^\text{ss}$ isomorphic over $\mathcal{H}$ to $\tilde{W}$. I claim that moreover $W$ is a simple constituent of $N_{\kappa^2}^\text{ss}$. To see this, note that since $\tilde{Q}_{\kappa,\text{red}}|_{Q_{\kappa^2,\text{red}}}$ and these are polynomials, there exists a positive integer $M$ such that $\tilde{Q}_{\kappa}|Q_{\kappa^2}^M$. Thus the fact that $\tilde{Q}_{\kappa}^*(U_{p^2})$ is zero on $\tilde{N}_{\kappa}$ (and therefore on $\tilde{W}$) implies that $Q_{\kappa^2}^*(U_{p}^M)$ is zero on $W$. But $Q_{\kappa^2}^*(U_{p})$ is invertible on $S_{\kappa^2}/N_{\kappa^2} \cong F_{\kappa^2}^*$, so $W$ must occur in $N_{\kappa^2}^\text{ss}$. Since $W$ and $\tilde{W}$ are isomorphic over $\mathcal{H}$, $h_{\kappa^2}U_{p}w = \alpha^{-1}w$ for all $w \in W \neq 0$, so $\alpha$ is a root of

$$\det(1 - hU_{p}T \mid N_{\kappa^2}^\text{ss}) = \det(1 - hU_{p}T \mid N_{\kappa^2}^\text{ss})$$

and the claimed divisibility follows. \hfill \Box

We now return to proving that the map (4) is well-defined. In other words, we must show that if $h$ is a polynomial over $A$ in the symbols $T_{\ell}$, $U_{\ell}$, and $(d)_{2N}$ that is nilpotent on $N$, and $\overline{h}$ is the corresponding polynomial over $B$ as usual, then $\overline{h}$ is nilpotent on $\overline{N}$. But if $h$ is nilpotent on $N$ then $\det(1 - hU_{p}T \mid N) = 1$, so by Lemma 5.7, $\det(1 - \overline{h}U_{p^2}T \mid \overline{N}) = 1$ as well. It follows that $\overline{h}U_{p^2}$ is nilpotent on $\overline{N}$ since this module is projective of finite rank, and hence so is $\overline{h}$ since $U_{p^2}$ acts invertibly on $\overline{N}$.

The (nonempty) admissible opens $g^{-1}(Y_{\text{red}})$ form an admissible cover of $\tilde{D}_{X,\text{red}}$, so in order to glue the maps we have defined to a map $\tilde{D}_{X,\text{red}} \longrightarrow D_{X,\text{red}}$ we must check that they agree on the overlaps. Since $2 : \tilde{X} \longrightarrow X$ is an isomorphism, this is immediate from the characterization on points afforded by the definition (4) and Lemma 3.3 since these spaces are reduced. Now note that, for each $i$, the spaces $W^{1}$ and $W^{2i}$ are covered by the nested families of affinoids $\{W^{1}_{n}\}$ and $\{W^{2i}_{n}\}$, respectively, and $2 : W^{1}_{n} \longrightarrow W^{2i}_{n}$ is an isomorphism for each $n$. Thus we may glue over increasing $n$ to obtain a diagram

$$\begin{array}{ccc}
\tilde{D}^{i}_{\text{red}} & \xrightarrow{\text{Sh}} & D^{2i}_{\text{red}} \\
\downarrow & & \downarrow \\
\tilde{Z}^{i}_{\text{red}} & \xrightarrow{g} & Z^{2i}_{\text{red}} \\
\downarrow & & \downarrow \\
W^{i} & \xrightarrow{2} & W^{2i}
\end{array}$$

where the superscript $i$ on a space mapping to $W$ denotes the preimage of the connected component $W$. Finally, since $W$ is the disjoint union of the $W^{i}$ we obtain the desired diagram (3).

We now come to the main result of this paper.

Theorem 5.8. — Let $N$ be a positive integer and let $p$ be an odd prime not dividing $N$. Let $D$ and $Z$ denote the integral weight cuspidal eigencurve and spectral curve of level $2N$, respectively, and let $\tilde{D}$ and $\tilde{Z}$ denote the half-integral weight cuspidal
eigencurve and spectral curve of level $4N$, respectively. There exists a unique diagram

$$
\begin{array}{c}
\tilde{D}_{\text{red}} \xrightarrow{\text{Sh}} D_{\text{red}} \\
\downarrow \hspace{1cm} \downarrow \\
\tilde{Z}_{\text{red}} \xrightarrow{g} Z_{\text{red}} \\
\downarrow \hspace{1cm} \downarrow \\
\mathcal{W} \xrightarrow{2} \mathcal{W}
\end{array}
$$

where $2$ and $g$ are characterized on points by

$$2(\kappa) = \kappa^2 \quad \text{and} \quad g(\kappa, \alpha) = (\kappa^2, \alpha)$$

and $\text{Sh}$ has the property that if $x \in \tilde{D}(L)$ corresponds to a system of eigenvalues occurring on a nonzero classical form $F \in \tilde{S}_k^{cl}(4Np, K, \chi^j)$ with $k \geq 5$, then $\text{Sh}(x)$ corresponds to the system of eigenvalues associated to the image of $F$ under the classical Shimura lifting.

Proof. — That the maps $\text{Sh}$ and $g$ that we have constructed have the indicated properties is clear from their construction and Theorem 2.5. Uniqueness follows from Lemma 4.5 since these spaces are reduced (the omission of the finitely many classical weights with $\lambda \leq 1$ does not affect the density of the classical points).

6. Properties of $\text{Sh}$

In this section we determine the nature of the image and fibers of the map $\text{Sh}$. For each $i$, the map $2: \mathcal{W}^i \rightarrow \mathcal{W}^{2i}$ is an isomorphism, and it follows easily from Lemma 3.3 and the definition (4) that the restriction $\text{Sh}: \tilde{D}^i_{\text{red}} \rightarrow D^{2i}_{\text{red}}$ is injective.

Proposition 6.1. — The map $\text{Sh}$ carries $\tilde{D}^i_{\text{red}}$ isomorphically onto a union of irreducible components of $D^{2i}_{\text{red}}$.

Proof. — Let $X \subseteq \mathcal{W}^{2i}$, $\tilde{X} \subseteq \mathcal{W}^i$, and $Y$ be as in Lemma 5.6. Then by that lemma $g^{-1}(Y_{\text{red}})$ is either empty or in $\mathcal{C}(\tilde{Z}_{\tilde{X}_{\text{red}}})$. Accordingly, $\text{Sh}^{-1}(D(Y)_{\text{red}})$ is either empty or equal to $\tilde{D}(\tilde{Y})_{\text{red}}$, where $\tilde{Y}$ is the element of $\mathcal{C}(\tilde{Z}_{\tilde{X}})$ of which $g^{-1}(Y_{\text{red}})$ is the underlying reduced affinoid. In the latter case, the map

$$\text{Sh}^*: T(Y)_{\text{red}} \rightarrow \tilde{T}(\tilde{Y})_{\text{red}}$$

is clearly surjective from its definition (4). Thus the image of $\text{Sh}$ is locally cut out by a coherent ideal (since the rings $T(Y)$ are Noetherian), and is therefore an analytic set in $D^i_{\text{red}}$. Since these spaces are reduced, $\text{Sh}$ is an isomorphism of $\tilde{D}^i_{\text{red}}$ onto this analytic set. Both $\tilde{D}^i_{\text{red}}$ and $D^i_{\text{red}}$ are equidimensional of dimension one by Lemma 5.8 of [1], so Corollary 2.2.7 of [5] ensures that the image of $\text{Sh}$ is a union of irreducible components.

\qed
Note that for each \(i\), there are exactly two connected components of \(W\) that map via \(2\) to \(W^{2i}\), namely \(W^i\) and \(W^i'\) where \(i' = i + (p - 1)/2\). We will construct a canonical isomorphism \(\tilde{D}^i \cong \tilde{D}^{i'}\) fitting into a commutative diagram

\[
\begin{array}{ccc}
\tilde{D}^i_{\text{red}} & \xrightarrow{\sim} & D^{2i}_{\text{red}} \\
\downarrow & & \downarrow \\
\tilde{D}^i'_{\text{red}} & \xrightarrow{\sim} & D^{2i'}_{\text{red}}
\end{array}
\]

In particular, it will follow that the diagonal arrows have the same union of irreducible components as image, and the map \(\text{Sh}\) is everywhere two-to-one. The existence of such an isomorphism stems from the existence of a Hecke operator \(U_p\) on families of overconvergent forms that is a kind of “square-root” of the operator \(U_p^2\).

In [10] we constructed operators \(U_\ell\) (on the spaces considered in that paper) having the effect \(\sum a_n q^n \mapsto -\sum a_{\ell n} q^n\) on \(q\-expansions, for all primes \(\ell\) dividing the level. These operators were found to commute with all other Hecke operators, but only commute with the diamond operators up to a factor of the quadratic character \((\ell/\cdot)\).

In our case, if \(\ell = p\), then such a map would in fact alter the weight since the \(p\)-part of the nebentypus is part of the \(p\)-adic weight character. Note that we have a factorization

\[
(p \cdot \cdot) = (\cdot \cdot) \cdot (\cdot \\
(p \cdot \cdot) = \cdot \cdot \\
\)

\[
\tau \cdot \cdot \cdot \\
\tau \cdot \cdot \\
\]

Let \(\epsilon\) denote the involution of \(W\) given by \(\epsilon(\kappa) = \kappa \cdot \tau(\cdot)\).

**Proposition 6.2.** — Fix a primitive \((4Np)^{th}\) root of unity \(\zeta_{4NP}\). Let \(X \subseteq W\) be a connected admissible affinoid open and let \(r \in [0, r_n] \cap \mathbb{Q}\). There is a compact \(\mathcal{O}(X)\)-linear map

\[
U_p : \hat{M}_{\text{c}(X)}(4N, \mathbb{Q}_p, p^{-r}) \otimes_{\mathcal{O}(\text{c}(X))} \mathcal{O}(X) \rightarrow \hat{M}_X(4N, \mathbb{Q}_p, p^{-r})
\]

having the effect \(\sum a_n q^n \mapsto \sum a_{pn} q^n\) on \(q\-expansions at (Tate(q), \zeta_{4NP})\). This map commutes with the operators \(T_\ell^2\) and \(U_\ell^2\) for all \(\ell\) and satisfies

\[
U_p \circ \langle d \rangle_{4N} = \left(\frac{\langle d \rangle_{4N}}{d} \right)^{(p - 1)/2} U_p \circ \langle d \rangle_{4N}.
\]

**Proof.** — The construction of \(U_p\), like the all of the operators \(T_\ell^2\) and \(U_\ell^2\) follows the general procedure set up in Section 5 of [9]. We will omit the details as they would take up considerable space and are very similar to the constructions of the operators \(T_\ell^2\) and \(U_\ell^2\), and content ourselves with commenting that the nontrivial commutation relation with the diamond operators arises (as it does in the construction of \(U_\ell\) given in [10]) because the “twisting” function \(H\) on \(X_1(4NP, p)_{\mathbb{Q}_p}^{an}\) used in the construction is not fixed by the diamond operators as it is in the case of \(T_\ell^2\) and \(U_\ell^2\). \(\square\)
Extending scalars to \( \mathcal{O}(\epsilon(X)) \) and replacing \( X \) by \( \epsilon(X) \), we arrive at a map in the opposite direction

\[
\tilde{M}_X(4N, \mathbb{Q}_p, p^{-r}) \hookrightarrow \tilde{M}_\epsilon(X)(4N, \mathbb{Q}_p, p^{-r}) \otimes_{\mathcal{O}(\epsilon(X))} \mathcal{O}(X)
\]

that has the same effect on \( q \)-expansions. It follows that the composition of these maps in either order is \( U_{p^2} \) (or its scalar extension). Everything we have said about \( U_p \) thus far holds equally well for cusps forms, and Lemmas 2.12 and 2.13 of \cite{1} now imply that

\[
\det(1 - U_{p^2}T | \tilde{S}_X) = \det(1 - (U_{p^2} \otimes 1)T | \tilde{S}_\epsilon(X) \otimes_{\mathcal{O}(\epsilon(X))} \mathcal{O}(X))
\]

\[
= \epsilon^* \det(1 - U_{p^2}T | \tilde{S}_\epsilon(X)).
\]

Since \( \epsilon \) is an isomorphism we get a diagram

\[
\begin{array}{ccc}
\tilde{Z}_X & \sim & \tilde{Z}_\epsilon(X) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\epsilon} & \epsilon(X)
\end{array}
\]

in which the horizontal arrows square to the identity map in the evident fashion. Note that this diagram establishes a bijection between the covers \( \mathcal{C}(\tilde{Z}_X) \) and \( \mathcal{C}(\tilde{Z}_\epsilon(X)) \). Using the links, we see that it is also compatible with enlarging \( X \) and hence one obtains an involution of the whole space \( \tilde{Z} \) covering the involution \( \epsilon \).

Let \( Y \in \mathcal{C}(\tilde{Z}_X) \) with connected image \( Y' \subseteq X \) and note that the corresponding element \( Y_\epsilon \in \mathcal{C}(\tilde{X}_\epsilon(X)) \) has connected image \( \epsilon(Y') \subseteq \epsilon(X) \). Corresponding to \( Y \) and \( Y_\epsilon \) we obtain decompositions

\[
\tilde{S}_X \otimes_{\mathcal{O}(X)} \mathcal{O}(Y') \cong \tilde{N} \oplus \tilde{F}
\]

and

\[
\tilde{S}_\epsilon(X) \mathcal{O}(\epsilon(X)) \mathcal{O}(Y') \cong \tilde{N}_\epsilon \oplus \tilde{F}_\epsilon
\]

respectively. Note that by extending scalars to \( \mathcal{O}(Y') \), \( U_p \) induces a map

\[
(\tilde{S}_\epsilon(X) \mathcal{O}(\epsilon(X)) \mathcal{O}(Y')) \otimes_{\mathcal{O}(X)} \mathcal{O}(Y') \rightarrow \tilde{S}_X \otimes_{\mathcal{O}(X)} \mathcal{O}(Y')
\]

**Lemma 6.3.** — This extension of scalars restricts to an isomorphism

\[
U_p : \tilde{N}_\epsilon \otimes_{\mathcal{O}(\epsilon(Y'))} \mathcal{O}(Y') \rightarrow \tilde{N}.
\]

**Proof.** — Let \( Q \in \mathcal{O}(Y')[T] \) be the polynomial factor of

\[
\det(1 - U_{p^2}T | \tilde{S}_X \otimes_{\mathcal{O}(X)} \mathcal{O}(Y'))
\]

associated to the choice of \( Y_\epsilon \) so that \( \epsilon^* Q \in \mathcal{O}(\epsilon(Y'))[T] \) is the polynomial associated to \( Y_\epsilon \). Thus the summand \( \tilde{N}_\epsilon \otimes_{\mathcal{O}(\epsilon(Y'))} \mathcal{O}(Y') \) of

\[
(\tilde{S}_\epsilon(X) \mathcal{O}(\epsilon(X)) \mathcal{O}(Y')) \otimes_{\mathcal{O}(X)} \mathcal{O}(Y')
\]

is precisely the kernel of

\[
\epsilon^*(\epsilon^* Q^*(U_{p^2})) = Q^*(U_{p^2})
\]
and since the map $U_p^2$ commutes with the action of $U_p$, this summand maps to the kernel of $Q^*(U_p^2)$ in $\tilde{S}_X \otimes \mathcal{O}(Y')$ under $U_p$. The latter is simply $\tilde{N}$, so $U_p$ at least restricts to some map

$$\tilde{N} \otimes \mathcal{O}(\epsilon(Y')) \to \tilde{N}.$$

As above we can extend scalars to $\mathcal{O}(\epsilon(Y'))$ and reverse the roles of $Y'$ and $\epsilon(Y')$ to get a map in the other direction with the property that both compositions are simply $U_p^2$ (or a scalar extension thereof) on the respective spaces. That these maps are isomorphisms now follows from the fact that $U_p^2$ is invertible on the modules $\tilde{N}$ and $\tilde{N}_e$.

It follows from Lemma 6.3 and the commutation relations in Proposition 6.2 that the map

$$\tilde{T}(\tilde{Y}) \to \tilde{T}(Y)$$

given by $\epsilon^*: \mathcal{O}(\epsilon(Y')) \to \mathcal{O}(Y')$ on coefficients and

$$(8) \quad T_{\ell^2} \mapsto T_{\ell^2},$$

$$U_{\ell^2} \mapsto U_{\ell^2},$$

$$\langle d \rangle_{4N} \mapsto \left(\frac{-1}{d}\right)^{(p-1)/2} \langle d \rangle_{4N}$$

on the generators is in fact well-defined. Thus we obtain a map $\tilde{D}_Y \to \tilde{D}_{Y_e}$ covering the map $Y \to Y_e$. This construction readily glues over $Y \in \mathcal{E}(\tilde{Z}_X)$. Moreover, the maps $U_p$ are compatible with the canonical links used in the construction of $\tilde{D}$ (because these links are simply induced by restriction to smaller admissible opens in $X_1(4NpK)$), and it follows that these maps glue to a map $\tilde{D}^i \to \tilde{D}^i$. When this map is composed with the one obtained by reversing the roles of $i$ and $i'$ (in either direction), one obtains the identity, as is evident from the definition (8). In particular it is an isomorphism and extends to an involution of the whole space $\tilde{D}$. Finally, that the diagram (7) commutes can be checked on points since these spaces are reduced. But then it follows immediately from the characterization of these points in terms of systems of eigenvalues in Lemma 3.3 and the definition (8).

**Example 6.4.** — The restriction $k \geq 5$ in Theorem 1.1 is there to avoid certain theta functions that show up in weights $1/2$ and $3/2$. There is no meaningful modular lifting of the theta series of weight $1/2$ (but see [3]). However, one can lift cuspidal theta functions of weight $3/2$, but one obtains Eisenstein series of weight $2$ instead of cusp forms. Let $\psi$ be a primitive Dirichlet character modulo a positive integer $r$ such that $\psi(-1) = -1$. Then

$$\theta_\psi(q) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \psi(n) nq^{n^2}$$

is a classical cusp form of weight $3/2$, level $4r^2$ and nebentypus character $\psi_1 = \psi \cdot (-1/\cdot)$. The form $\theta_\psi$ is an eigenform for all $T_{\ell^2}$ ($\ell \nmid 4r^2$) and $U_{\ell^2}$ ($\ell \mid 4r^2$) with
eigenvalues \((1 + \ell)\psi(\ell)\). The Shimura lift of this form is the Eisenstein series

\[
E_\psi(q) = \sum_{n=1}^{\infty} \psi(n)\sigma(n)q^n
\]

of weight 2, where \(\sigma(n) = \sum_{d|n} d\), as it easy to check from the explicit formulas in [11].

Let \(p\) be an odd prime. If \(p \mid r\), then \(\theta_\psi\) is in the kernel of \(U_p^2\) and does not furnish a point on the eigencurve \(\tilde{D}\). If on the other hand \(p \nmid r\), then \(\theta_\psi\) thought of in level \(4r^2p\) is in fact a \(U_p^2\) eigenform with eigenvalue \(\psi(p)p \neq 0\). Thus \(\theta_\psi\) furnishes a point on \(\tilde{D}\) and the existence of the map \(\text{Sh}\) implies that there exists a cuspidal \(p\)-adic eigenform with the same eigenvalues. It is easy to check that the form

\[
E_\psi^* = E_\psi - \psi(p)V_p E_\psi
\]

has the expected eigenvalues, so this must be the image form. In fact it is easy to check using the explicit formulas in [11] that this is in fact the classical Shimura lift applied to \(\theta_\psi\) thought of at level \(4r^2p\).

Thus we a lead to the conclusion that \(E_\psi^*\), while not a classical cusp form, is a cuspidal \(p\)-adic modular form. That is, \(E_\psi^*\) vanishes at the cusps in the connected component \(X_1(4r^2p)^{\geq 1}\) of the ordinary locus. This fact also follows from Theorem 3.4 of [3], where Cipra computes the value of \(E_\psi^*\) at every cusp. We also remark that this is only possible since \(E_\psi^*\) is of critical slope (in this case, slope 1) since if it were of low slope then the technique of Kassaei [7] implies that a low-slope form that is \(p\)-adically cuspidal is in fact a classical cuspidal modular form.

References


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