NON-UNIQUENESS FOR SQUARE CONVERGENT DOUBLE TRIGONOMETRIC SERIES?

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Abstract. If a multiple trigonometric series converges everywhere to zero in the sense of spherical convergence, of unrestricted rectangular convergence, or of iterated convergence, then that series must have every coefficient being zero. But the cases of square convergence and restricted rectangular convergence lead to open questions, even in dimension 2.

There are several obstacles to proving a uniqueness theorem for double trigonometric series for the two methods, square convergence and restricted rectangular convergence. First of all, there is no Cantor-Lebesgue theorem, at least in the usual sense. What I mean by this is that the usual Cantor-Lebesgue theorem associated with one parameter methods of convergence assumes that the difference of successive partial sums tends to zero everywhere (or at least at every point of a “substantial” subset of $T^d$), and concludes that the coefficients themselves must be small. But there is a double trigonometric series which is square convergent to a finite value at every point, so that the differences of its partial sums tend to zero everywhere, whose coefficients grow faster than any polynomial. The series is

\[ T(x) = 2\sqrt{\pi} \sum_{n=2}^{\infty} n^{3/2} e^{n/x} \cos^2 \left( \frac{x}{2} \right) \sin^{2n-2} \left( \frac{x}{2} \right) \cos ny. \]

The partial sums of the series as written coincide with the square partial sums of a double trigonometric series as can be seen from expanding

\[ \cos^2 \left( \frac{x}{2} \right) \sin^{2n-2} \left( \frac{x}{2} \right) = \left( \frac{e^{ix/2} + e^{-ix/2}}{2} \right)^2 \left( \frac{e^{ix/2} - e^{-ix/2}}{2i} \right)^{2n-2}. \]

If $x = \pi$ or $-\pi$, $\cos \left( \frac{x}{2} \right) = 0$ so $T(-\pi) = T(\pi) = 0$, while if $x \in (-\pi, \pi)$, then $a = \sin^2 \left( \frac{x}{2} \right) < 1$, so that the series defining $T$ converges absolutely by comparison with $2\sqrt{\pi}/a \sum_n n^{3/2} e^{n/\ln a} a^n$. To see that the $(0,n)$ coefficients of $T$ grow like $e^{n/\ln n}$ involves multiplying out the right hand side of equation (0.2). It is also true, but somewhat more technical to verify, that this series is also restrictedly rectangularly convergent to a finite value at every point. See [AWa2] for details.

It is logically possible that everywhere square convergence to zero is different than everywhere convergence to finite values, but this would involve a completely new and much more delicate type of Cantor-Lebesgue type theorem. So the method of proof used for uniqueness theorems for other methods of convergence by Cantor/C,
A], Shapiro[Con, Coo,S], and Bourgain[B, AWa,A1], which involves forming a Riemann function or second integral of the original series by dividing by \( |n|^2 \) is not very likely to have an analogue here.

Second, I have devoted some effort to trying to prove a conditional square uniqueness theorem in the spirit of Shapiro’s work. In other words, I simply add some reasonably mild condition on the coefficients such as
\[ c_{mn} \to 0 \text{ as } \max \{|m|,|n|\} \to \infty. \]
Even such a conditional uniqueness theorem for square convergence seems difficult to achieve at this time.

With all efforts to prove something positive at a standstill, it seems natural to wonder if there might be a counterexample. Since at a fixed point restricted rectangular convergence implies square convergence, proving a uniqueness theorem should be easier for restricted rectangular convergence, while finding a counterexample to uniqueness should be easier for square convergence. So we will move in the direction of trying to find a counterexample to square uniqueness; that is, of trying to construct a double trigonometric series that is square convergent to 0 everywhere.

Conjecture 1. There is a nontrivial double trigonometric series that is square convergent to 0 at every point of \([0,2\pi]^2\).

We start with an example of a one dimensional trigonometric series that has a subsequence of partial sums that converges to zero everywhere. Such a series was discovered by Kozlov.[K] See example 2.2 on pages 187–190 of[AWa1] for two different ways to construct such a sequence. The basic fact used amounts to this.

**FACT.** Given any trigonometric polynomial \( f = \sum_{n=1}^{N} a_n \sin nx \), any \( \varepsilon > 0 \), any \( \eta > 0 \), and any integer \( M > N \), there can be found a trigonometric polynomial \( p = \sum_{n=M}^{M+R} a_n \sin nx \) such that for all \( x \in \mathbb{T} \setminus (-\varepsilon, \varepsilon) \), \( |f(x) + p(x)| < \eta \).

We will construct two one dimensional trigonometric series, \( P(x) = p_1(x) + p_2(x) + \ldots \), and \( Q(y) = q_1(y) + q_2(y) + \ldots \) where every \( p \) and \( q \) is a linear combination of sine functions, the lowest frequency of each \( p_{n+1} \) is greater than the highest frequency of \( p_n \), and the \( q \)'s have the same property. We will then consider the resulting double trigonometric series
\[ T(x, y) = P(x) Q(y). \]

The \( n \)th square partial sum of \( T \) is exactly the product of the \( n \)th partial sum of \( P \) and the \( n \)th partial sum of \( Q \),
\[ T_{nn}(x, y) = P_n(x) Q_n(y). \]

The plan is to design \( P \) and \( Q \) is such a way that for \( (x, y) \) fixed, for a certain subsequence of \( n \), \( P_n \) tends to zero and \( Q_n \) is not too big, while for the subsequence consisting of the remaining \( n \), \( P_n \) is not too big and \( Q_n \) tends to zero.

Let \( \{\varepsilon_i\} \) be a sequence of positive numbers tending monotonically to 0. Then the intervals \( \{(-\varepsilon_i, \varepsilon_i)\} \) shrink to 0, so the complementary subintervals of \( \mathbb{T} \), \( I_i = [-\pi, -\varepsilon_i] \cup [\varepsilon_i, \pi] \) increase monotonically to \( \mathbb{T} \setminus \{0\} \). Also let \( \{\eta_n\} \) be another sequence of positive numbers tending monotonically to 0. Start with \( p_1(x) = \eta_1 \sin x \).
so that
\[
\sup_{x \in I_1} |p_1(x)| \leq \eta_1
\]
and let \(m_1 = \deg p_1 = 1\). Then use the FACT to pick \(p_2\) of degree \(m_2\) with frequencies starting at \(m_1 + 1 = 2\) so that \(p_2\) satisfies
\[
\sup_{x \in I_2} |p_1(x) + p_2(x)| \leq \eta_2
\]
This creates a first “bad \(x\) zone,” [2, \(m_2 - 1\)], bad in the sense that for \(n\) in this interval, the \(n\)th partial sum of \(P\) may not be small. So, let \(q_1\) be a nontrivial trigonometric polynomial in \(y\) of degree \(n_1 = 1\) satisfying
\[
\sup_{y \in I_1} |q_1(y)| \leq \eta_1,
\]
in particular \(q_1(y) = \eta_1 \sin y\). Next use the FACT to pick \(q_2(y)\) to have frequencies starting with \(m_2\), to be of degree \(n_2\), and to satisfy
\[
\sup_{y \in I_2} |q_1(y) + q_2(y)| \leq \eta_2.
\]
We have
\[
T_{11}(x, y) = p_1(x)q_1(y) = \eta_1^2 \sin x \sin y
\]
and if \(n\) is in the first bad \(x\) zone,
\[
T_{nn}(x, y) = (p_1(x) + p_2^*(x))(q_1(y)),
\]
where \(p_2^*\) is a partial sum of \(p_2\). This creates a first bad \(y\) zone; for \(n \in [m_2, n_2 - 1]\), the \(n\)th partial sum of \(Q\) may not be small. For \(n\) in the first bad \(y\) zone,
\[
T_{nn}(x, y) = (p_1(x) + p_2(x))(q_1(y) + q_2^*(y)),
\]
where \(p_2^*\) is a partial sum of \(p_2\).

Now use the FACT to pick \(p_3\) to have frequencies starting with \(n_2\), to be of degree \(m_3\), and to satisfy
\[
\sup_{x \in I_3} |p_1(x) + p_2(x) + p_3(x)| \leq \eta_3.
\]
The second bad \(x\) zone is \([n_2, m_3 - 1]\) and for \(n\) in this zone, we have
\[
T_{nn}(x, y) = (p_1(x) + p_2(x) + p_3^*(x))(q_1(y) + q_2(y)),
\]
where \(p_3^*\) is a partial sum of \(p_3\). So use the FACT to pick \(q_3\) to have frequencies starting with \(m_3\), to be of degree \(n_3\), and to satisfy
\[
\sup_{y \in I_3} |q_1(y) + q_2(y) + q_3(y)| \leq \eta_3
\]
This creates a second bad \(y\) zone, \([m_3, n_3 - 1]\) on which the \(n\)th partial sum of \(Q\) may not be small.

We continue inductively. Having chosen \(p_{k-1}\) with frequencies belonging to \([n_{k-2}, m_{k-1}]\) and satisfying
\[
\sup_{x \in I_{k-1}} |p_1(x) + \cdots + p_{k-1}(x)| \leq \eta_{k-1}
\]
and also \(q_{k-1}\) with frequencies belonging to \([m_{k-1}, n_{k-1}]\) and satisfying
\[
\sup_{x \in I_{k-1}} |q_1(y) + \cdots + q_{k-1}(y)| \leq \eta_{k-1},
\]
use the FACT to choose first \( p_k \) with frequencies belonging to \([n_{k-1}, m_k]\) and satisfying
\[
\sup_{x \in I_k} |p_1(x) + \cdots + p_k(x)| \leq \eta_k
\]
and then use the FACT to choose \( q_k \) with frequencies belonging to \([m_k, n_k]\) and satisfying
\[
\sup_{x \in I_k} |q_1(y) + \cdots + q_k(y)| \leq \eta_k.
\]
Notice that if \( n \) is in the \((k-1)\)th bad \(x\) zone \([n_{k-1}, m_k-1]\), then
\[
T_{nn} (x, y) = (p_1(x) + \cdots + p_k(x)) (q_1(y) + \cdots + q_{k-1}(y)),
\]
where \( p_k^* (x) \) is a partial sum of \( p_k(x) \), while if \( n \) is in the \((k-1)\)th bad \(y\) zone \([m_k, n_k-1]\), then
\[
T_{nn} (x, y) = (p_1(x) + \cdots + p_k(x)) (q_1(y) + \cdots + q_k(y)),
\]
where \( p_k^* (x) \) is a partial sum of \( p_k(x) \).

We remark that this construction has been carried out in such a way that the partial sums of
\[
P(x) = p_1(x) + \cdots + p_k(x) + \cdots
\]
have the constant value
\[
P_m(x) = p_1(x) + \cdots + p_k(x)
\]
for \( m = m_k, m_k + 1, \ldots, n_k - 1 \), and the partial sums of \( Q(y) \) have the constant value
\[
Q_n(y) = q_1(y) + \cdots + q_k(y)
\]
for \( n = n_k, n_k + 1, \ldots, m_{k+1} - 1 \).

Let \((x, y)\) be any point of \(\mathbb{T}^2\). If \(x = 0\), then \(P_n(x) = 0\) for all \(n\), so that
\[
\lim_{n \to \infty} T_{nn} (0, y) = \lim_{n \to \infty} 0 \cdot Q_n (y) = 0.
\]
Similarly,
\[
\lim_{n \to \infty} T_{nn} (x, 0) = \lim_{n \to \infty} P_n (x) \cdot 0 = 0.
\]
The question is whether the polynomials \( \{p_m\} \) and \( \{q_n\} \) can be chosen in such a way that for every other pair \((x, y)\) \(\in \mathbb{T}^2\),
\[
\lim_{n \to \infty} T_{nn} (x, y) = 0.
\]

The basic idea of the construction is that every square partial sum of the double series \(T\) is a product of two terms and one of these two terms is always very small. The hope for constructing a counterexample to square uniqueness lies in trying to control the other term.

**Conjecture 2.** In the above construction it is possible to pick the sequence \( \{\eta_n\} \searrow 0 \) and the trigonometric polynomials \( \{p_m\} \) and \( \{q_n\} \) in such a way that for fixed nonzero \(x\) and \(y\),
\[
\lim_{k \to \infty} \left( \sup_{\ell} |p_{k,\ell}(x)| \right) \eta_{k-1} = 0
\]
and also
\[
\lim_{k \to \infty} \left( \sup_{\ell} |q_{k,\ell}(y)| \right) \eta_k = 0,
\]
where \( p_{k,\ell} \) and \( q_{k,\ell} \) denote the \(\ell\)th partial sums of \( p_k \) and \( q_k \).
Notice that the process of picking the $m_i$'s and $n_i$'s is such that
\[ 1 = m_1 = n_1 < m_2 < n_2 < m_3 < n_3 < \cdots < m_{k-1} < n_{k-1} < m_k < n_k < \cdots \]
so that every index $n \geq 2$ is either in a bad $x$ zone (when $n_{k-1} \leq n \leq m_k - 1$ for some $k$) or a bad $y$ zone (when $m_k \leq n \leq n_k - 1$ for some $k$).

Now fix $(x, y)$ with both $x$ and $y$ not zero. If $n$ is sufficiently large, either it is in $[n_{k-1}, m_k - 1]$ or $[m_k, n_k - 1]$ for a $k$ with the property that both $x$ and $y$ are in $I_k$. In the former case, from equation (0.3) we have the estimate
\[
|T_{nn}(x, y)| = |p_1(x) + \cdots + p_k(x)| |q_1(y) + \cdots + q_k(y)| \\
\leq \left( \eta_{k-1} + \sup_{\ell} |p_{k,\ell}(x)| \right) \eta_{k-1} \\
= o(1) + \left( \sup_{\ell} |p_{k,\ell}(x)| \right) \eta_{k-1},
\]
while in the latter case from equation (0.4) we have the similar estimate
\[
|T_{nn}(x, y)| = |p_1(x) + \cdots + p_k(x)| |q_1(y) + \cdots + q_k(y)| \\
\leq \eta_k \left( \eta_{k-1} + \sup_{\ell} |q_{k,\ell}(x)| \right) \\
= o(1) + \left( \sup_{\ell} |q_{k,\ell}(y)| \right) \eta_k.
\]

From these two estimates it is immediate that if the conjecture can be satisfied, then the double trigonometric series
\[ T(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_m c_n \sin mx \sin ny \]
is everywhere on $\mathbb{T}^2$ square convergent to zero. This would violate uniqueness for square convergence. Unfortunately, both proofs of the FACT, while potentially constructive, have so far only been carried out in a nonconstructive way, so that while it is clear what has to be done to test the conjecture, I have not had the courage to try it.

One thing that can be stated in favor of this program is that it is not ruled out by virtue of producing a counterexample to either the unrestricted rectangular uniqueness Theorem [AFR, Awe] nor the iteratedly uniqueness Proposition [Tet] discussed above. Indeed, we can show the following proposition.

**Proposition 1.** A double trigonometric series of the form $T(x, y) = P(x)Q(y)$ where $P$ and $Q$ are nontrivial trigonometric series is neither iteratively nor unrestrictedly rectangularly convergent to zero everywhere.

**Proof.** Since $P$ is nontrivial,
\[ B = \{ x \in I : \{ P_m(x) \} \text{ does not tend to 0 at } x \} \]
is nonempty, for otherwise Cantor’s one dimensional nonuniqueness theorem would be violated. Fix any $x \in B$. There is an extended real number $s \neq 0$ (may be $+\infty$ or $-\infty$) and a sequence $\{ \mu_k \}$ such that
\[
\lim_{j \to \infty} P_{\mu_j}(x) = s.
\]
Similarly, let
\[ C = \{ y \in \mathbb{T} : \{ Q_n(y) \} \text{ does not tend to } 0 \text{ at } y \}, \]
fix a \( y \in C \), and find an extended real number \( t \neq 0 \) and a sequence \( \{ \nu_k \} \) such that
\[
(0.6) \quad \lim_{k \to \infty} Q_{\nu_k}(y) = t.
\]
Then
\[
(0.7) \quad \lim_{j,k \to \infty} P_{\mu_j}(x) Q_{\nu_k}(y) = st.
\]
For \( T(x,y) \) to be iteratively convergent to zero at the point \((x,y)\), we must have both
\[
\lim_{m \to \infty} \left( \lim_{n \to \infty} P_m(x) Q_n(y) \right) = 0,
\]
and
\[
\lim_{n \to \infty} \left( \lim_{m \to \infty} P_m(x) Q_n(y) \right) = 0.
\]
Actually, at the point \((x,y)\) neither limit is \(0\). By symmetry, it is enough to see that the first limit is not zero. Suppose it were zero. Then we would have
\[
0 = \lim_{m \to \infty} \left( \lim_{n \to \infty} P_m(x) Q_n(y) \right) = \lim_{m \to \infty} \left( P_m(x) \lim_{n \to \infty} Q_n(y) \right) = \left( \lim_{m \to \infty} P_m(x) \right) \left( \lim_{n \to \infty} Q_n(y) \right).
\]
Since subsequential limits must agree with limits, it would follow that
\[
(0.8) \quad 0 = \left( \lim_{j \to \infty} P_{\mu_j}(x) \right) \left( \lim_{k \to \infty} Q_{\nu_k}(y) \right).
\]
But, by relation (0.6),
\[
t = \lim_{k \to \infty} Q_{\nu_k}(y),
\]
and by relation (0.5),
\[
s = \lim_{j \to \infty} P_{\mu_j}(x),
\]
which contradicts equation (0.8), since \( st \neq 0 \). Similarly, if \( T \) were unrestrictedly rectangularly convergent to zero at \((x,y)\), in particular we would have
\[
0 = \lim_{k \to \infty} T_{\mu_k,\nu_k}(x,y) = \lim_{k \to \infty} P_{\mu_k}(x) Q_{\nu_k}(y),
\]
which is contrary to relations (0.5) and (0.6).

References


G. Cantor, *Beweis, das eine für jeden reellen Wert von \( x \) durch eine trigonometrische Reihe gegebene Funktion \( f(x) \) sich nur auf eine einzige Weise in dieser Form darstellen lässt*, Crelles J. für Math., 72(1870), 139–142; also in Gesammelte Abhandlungen, Georg Olms, Hildesheim, 1962, pp. 80–83.


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