GENERALIZATIONS OF THE RIEMANN DERIVATIVE[1]

BY

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Introduction. In §1 of this paper a derivative generalizing the Riemann derivative is considered. The existence of this derivative on a set is shown to imply the existence of the Peano derivative almost everywhere on the set. In §2 the L^p norm $(1 \leq p < \infty)$ replaces the L^∞ norm of §1 and the same result is proved. A special case of this result is that the existence of the Riemann L^p derivative implies the existence of the Peano L^p derivative almost everywhere. In §3 a generalization of smoothness is shown to imply smoothness almost everywhere. We consider only measurable sets of real numbers and real valued functions of a real variable.

1. **An L^∞ generalization of the Riemann derivative.** A function f is said to have a Peano derivative of order k at x, i.e., $f \in t_k(x)$, if there are constants $f_0(x)$, $f_1(x), \ldots, f_k(x)$ such that

$$f(x+t) = f_0(x)+f_1(x)t+\cdots+\frac{f_k(x)}{k!}t^k+o(t^k) \qquad \text{as } t \to 0.$$

We say f is Peano bounded of order k at x, i.e., $f \in T_k(x)$, if there are constants $f_0(x), \ldots, f_{k-1}(x)$ such that

$$f(x+t) = f_0(x)+f_1(x)t+\cdots+\frac{f_{k-1}(x)}{(k-1)!}t^{k-1}+O(t^k) \qquad \text{as } t \to 0.$$

Let $A=\{a_0, a_1, \ldots, a_{k+l}; A_0, \ldots, A_{k+l}\}$ be a set of real numbers with $a_i \neq a_j$ if $i \neq j$ satisfying

$$\begin{aligned} \sum_{i=0}^{k+l} A_i a_i^j &= 0, \qquad j = 0, 1, \ldots, k-1, \\ &= k!, \qquad j = k. \end{aligned}$$

We say that f has a kth generalized derivative with respect to A at the point x, i.e., $f \in g_k(x, A)$, if there is a constant $f_{(k)}(x) = f_{(k)}(x, A)$ such that

$$\sum_{i=0}^{k+l} A_i f(x+a_i t) = f_{(k)}(x)t^k+o(t^k) \qquad \text{as } t \to 0.$$

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A function f is generalized-bounded of order k with respect to A at x, i.e., $f \in G_k(x, A)$, if

$$\sum_{i=0}^{k+l} A_i f(x+a_i t) = O(t^k) \qquad \text{as } t \to 0.$$

To demonstrate the reason for the conditions on $\sum_{i=0}^{k+l} A_i a_i^j$, let $f \in t_k(x)$. Then

$$\begin{aligned}\sum_{i=0}^{k+l} A_i f(x+a_i t) &= \sum_{i=0}^{k+l} A_i \left[\sum_{j=0}^{k} \frac{f_j(x)}{j!}(a_i t)^j + o(t^k)\right] \\ &= \sum_{j=0}^{k} \frac{f_j(x)}{j!} t^j \left[\sum_{i=0}^{k+l} A_i a_i^j\right] + o(t^k) \\ &= f_k(x) t^k + o(t^k) \qquad \text{as } t \to 0.\end{aligned}$$

In other words, the conditions assure that if the Peano derivative exists, the generalized derivative will exist and be equal to it.

If $l=0$ and if the a_i's are given, since the $k+1$ A_i's must satisfy the $k+1$ conditions, and since the matrix $((a_i^j))$ is a Van der Monde matrix and hence invertible, it follows that the A_i's can be expressed in terms of the a_i's. To be precise (see Denjoy [1])

$$A_i = \left[\prod_{j \neq i} (a_i - a_j)\right]^{-1} \cdot k! \qquad i = 0, 1, \dots, k.$$

If, on the other hand, $l > 0$, the a_i's and the $k+1$ conditions do not uniquely determine the A_i's. l will be called the *excess*.

Probably the most important example of the generalized derivative is the Riemann derivative. The kth Riemann derivative is obtained by setting

$$a_i = -\frac{k}{2} + i, \qquad i = 0, 1, \dots, k.$$

Since $l=0$, we find that

$$A_i = \left(\prod_{j \neq i}\left[\left(-\frac{k}{2}+i\right) - \left(-\frac{k}{2}+j\right)\right]\right)^{-1} k! = \binom{k}{i}(-1)^{k-i}.$$

The relationships between the various derivatives which have been introduced may be displayed diagrammatically as at top of p. 183.

The arrows denote inclusion. For example, if $f \in t_k(x)$, then $f \in T_k(x)$, so that $t_k(x) \subseteq T_k(x)$. As may be shown by simple counterexample, none of the arrows may be reversed.

However, there is a classical theorem of Zygmund and Marcinkiewicz which states that if a function is k Riemann-bounded on a set, then at almost every point of that set it has a kth Peano derivative [2]. This may be generalized to

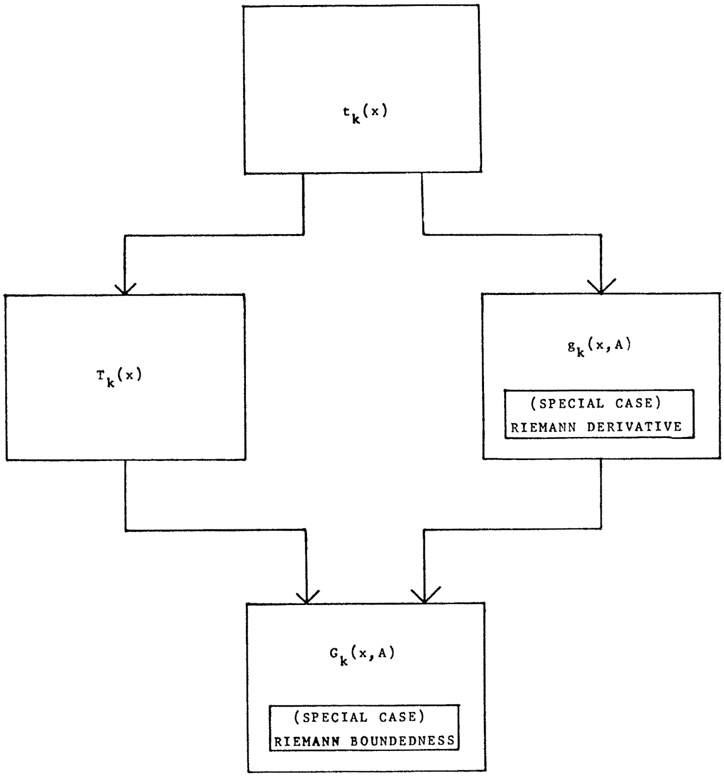

FIGURE 1 (Relationships between derivatives).

THEOREM 1. *Let $f \in G_k(x, A)$ for all $x \in E$, then $f \in t_k(x)$ for almost every $x \in E$.*

(The case $k=1$, $l=0$ is classical. The case $k=2$, $l=0$ was done by Marcinkiewicz and Zygmund [3].)

LEMMA 1. *Let 0 be a point of density of $\mathscr{E}$. Let $\{\alpha_i, \beta_i\}$, $i=1, \ldots, m$ be any set of real numbers such that $\beta_i \neq 0$ for all i. Then for all $u>0$ sufficiently small, there is a $v \in [u, 2u]$ such that*

$$\alpha_i u+\beta_i v \in \mathscr{E}, \qquad i = 1, \ldots, m.$$

Proof. Let φ be the characteristic function of $\mathscr{E}$. Let

$$A_i = \{v \in [u, 2u] \mid \alpha_i u+\beta_i v \in \mathscr{E}\}.$$

For each i,

$$|A_i| = \int_u^{2u} \varphi(\alpha_i u+\beta_i v)\, dv = \frac{1}{\beta_i} \int_{(\alpha_i+\beta_i)u}^{(\alpha_i+2\beta_i)u} \varphi(s)\, ds$$

$$\to \frac{1}{\beta_i} [(\alpha_i+2\beta_i)u-(\alpha_i+\beta_i)u] = u \qquad \text{as } u \to +0$$

since 0 is a point of density of $\mathscr{E}$. Hence by choosing u sufficiently small we can make

$$|A_i| > \left(1-\frac{1}{m}\right)u, \qquad i = 1, 2, \ldots, m.$$

For any sets A and B, let $A-B$ denote all the points of A that do not belong to B. Then

$$\left|[u, 2u]-\bigcap_{i=1}^{m} A_i\right| = \left|\bigcup_{i=1}^{m} ([u, 2u]-A_i)\right|$$
$$\leqq \sum_{i=1}^{m} |[u, 2u]-A_i| < \sum \frac{1}{m} u = u$$

so that

$$[u, 2u] \cap \left(\bigcap_{i=1}^{m} A_i\right) \neq \varnothing$$

and any v of this set will satisfy the conclusion of the lemma.

Lemma 2. *The sliding lemma: Suppose that $\alpha \geqq 0$, $n \geqq 1$ and*

$$\sum_{i=0}^{n} A_i f(x+a_i t) = O(t^{\alpha}) \qquad \textit{for all } x \in E,$$

then for any real a,

$$\sum_{i=0}^{n} A_i f(x+(a_i-a)t) = O(t^{\alpha}) \qquad \textit{for almost every } x \in E.$$

If "O" is replaced by "o" in the hypothesis, then the conclusion also holds with "o" in place of "O".

Proof. We may assume $a_0 \neq 0$ by reordering the terms if necessary. We may assume that $0<|E|<\infty$. Let $E_j=\{x \in E \mid |\sum_{i=0}^{n} A_i f(x+a_i t)| \leqq j|t|^{\alpha}$ if $|t|<1/j\}$. Since $|E-E_j| \to 0$, it suffices to prove the lemma at every point of E_j which is a point of density of E_j. To simplify notation, let $x=0$ be such a point. Let t be greater than 0 (the case $t<0$ is proved similarly). By Lemma 1, if t is sufficiently small, there is a $u \in [t, 2t]$ such that all of

$$(a_i-a)t-a_0 u, \qquad i = 0, 1, \ldots, n$$

and

$$-at+(a_k-a_0)u, \qquad k = 1, \ldots, n$$

belong to E_j. Since $(a_i-a)t-a_0 u \in E_j$, we have

$$\sum_{k=0}^{n} A_k f[\{(a_i-a)t-a_0 u\}+a_k u] = O(u^{\alpha}) = O(t^{\alpha}), \qquad i = 0, 1, \ldots, n.$$

Multiplying the ith equation by A_i and summing over i, we have

$$\sum_{i=0}^{n} A_i\left[\sum_{k=0}^{n} A_k f[\{(a_i-a)t-a_0u\}+a_ku]\right] = O(t^\alpha).$$

Rearranging the order of summation, we obtain

$$\sum_{k=0}^{n} A_k\left[\sum_{i=0}^{n} A_i f[\{-at+(a_k-a_0)u\}+a_it]\right] = O(t^\alpha).$$

Since the term in curly brackets is in E_j for $k>0$, each term of the outer sum except the $k=0$ term is $O(t^\alpha)$. Hence that term is also $O(t^\alpha)$, i.e.,

$$A_0\sum_{i=0}^{n} A_i f[\{-at+(a_0-a_0)u\}+a_it] = O(t^\alpha).$$

Dividing by A_0 and simplifying, we have

$$\sum_{i=0}^{n} A_i f[(a_i-a)t] = O(t^\alpha) \qquad \text{as } t\to+0,$$

which is the desired result. The "o" case is proved in a very similar manner.

LEMMA 3. *If* $|\sum_{i=0}^{n} A_i f(x+a_it)|=O(1)$ *for all* $x\in E$, *then* f *is bounded in a neighborhood of almost every point* $x\in E$.

We omit the proof which is similar to that of Lemma 5 on page 13 of [2].

Because of Lemma 3, without loss of generality we may add the assumption that f is bounded to the hypotheses of Theorem 1. Since f is bounded it is locally integrable; so we may define $D^{-i}f$ by

$$D^0f(x) = f(x),\ D^{-1}f(x) = \int_a^x f(t)\,dt,\ldots,\ D^{-i}f(x) = \int_a^x D^{-(i-1)}f(t)\,dt.$$

We come now to the cornerstone of the proof of Theorem 1.

LEMMA 4. *Suppose that* f *is bounded and*

$$\sum_{i=0}^{k+l} A_i f(x+a_it) = O(t^k) \qquad \text{as } t\to 0 \text{ for all } x\in E.$$

Then there is an integer $s\geqq 0$ *such that* $D^{-s}f$ *is* $(k+s)$ *Riemann-bounded at almost every* $x\in E$.

Proof. First let us suppose that all the a_i's are integers and that $l=0$. We may suppose that $a_0<a_1<\cdots<a_k$. From the sliding lemma it follows that we may assume $a_0=1$. If there are no gaps in the sequence $\{a_0,\ldots,a_k\}$, i.e., if $a_0=1$, $a_1=2$, $a_2=3,\ldots,a_k=k+1$, after sliding the sequence to the left by $k/2+1$, we deduce from the remarks preceding Figure 1 that f is k Riemann-bounded almost everywhere on E. If there are gaps, we fill them by integration. For example, if

$a_1 > 2$, we adjoin 2 to the set of a_i's as follows. Sliding the original derivative to the left by 2, we obtain

$$\sum_{i=0}^{k} A_i f(x+(a_i-2)t) = O(t^k) \qquad \text{for almost every } x \in E.$$

Now integrating from 0 to h where h is small, we obtain

$$\sum_{i=0}^{k} \frac{A_i}{a_i-2} D^{-1}f(x+(a_i-2)h) - \left[\sum_{i=0}^{k} \frac{A_i}{a_i-2}\right] D^{-1}f(x) = O(h^{k+1})$$

for almost every x in E.

Finally sliding this to the right by 2, we obtain

$$\sum_{i=0}^{k} \frac{A_i}{a_i-2} D^{-1}f(x+a_ih) - \left[\sum_{i=0}^{k} \frac{A_i}{a_i-2}\right] D^{-1}f(x+2h) = O(h^{k+1})$$

for almost every $x \in E$. This result shows that almost everywhere on E, $D^{-1}f$ is $k+1$ generalized-bounded with respect to a set whose a_i's have one fewer gap than had the original set of a_i's(2). Note that the excess is still 0. If $a_k = k+1+s$, there were s gaps initially, so after repeating this filling process $s-1$ more times, we obtain the conclusion of the lemma.

Next we suppose that the a_i's are integers, but that $l > 0$. Fix l. It suffices to show that for some positive integer s_1, $D^{-s_1}f$ is $k+s_1$ generalized-bounded with respect to a $(k+s_1)$th generalized derivative of excess $\leq l-1$. For if we can do this, an at most l-fold iteration of the process will reduce this case to the $l=0$ case above.

By employing the process of filling in the gaps, we may suppose that $a_0 = 1$, $a_1 = 2, \ldots, a_{k+l} = k+l+1$. It is important to note that the process of filling never increases the excess since at each step the order of the derivative is increased by one, while the number of a_i's is increased by at most one. (The process of filling may actually decrease the excess. For example, if $a_1 > 2$ and $\sum_{i=0}^{k+l} A_i/(a_i-2) = 0$, then after sliding to the left by 2, we have a $(k+1)$st derivative based on the original $k+l+1$ a_i's so that the excess is immediately $l-1$.)

Set $r = k+l+1$. Recalling that we may suppose $a_{i-1} = i$, $1 \leq i \leq r$, we may now write our assumption

$$\sum_{i=1}^{r} A_{i-1} f(x+it) = O(t^k) \qquad \text{as } t \to 0, \tag{1.1}$$

for all $x \in E$.

(2) If $\sum_{i=0}^{k+l} (A_i a_i^j)$ is equal to 0 when $j < k$, is equal to $k!$ when $j = k$, then $\sum_{i=0}^{k+l} A_i(a_i - a)^j = \sum_{i=0}^{k+l} (A_i a_i^j)$ so that any slide of a kth generalized derivative is still a kth generalized derivative. Also if no $a_i = 0$, $\sum_{i=0}^{k+l} (A_i/a_i)a_i^j$ is equal to $\sum_{i=0}^{k+l} (A_i/a_i)$ when $j = 0$, is equal to 0 when $j = 1, 2, \ldots, k$ and equals $k!$ when $j = k+1$ so that integration from 0 to h yields a $(k+1)$st generalized derivative. To obtain proper normalization, each integration should be coupled with a multiplication by the constant $k+1$. We shall always assume that this has been done.

Sliding this to the left by $r+1$, we have

$$\sum_{i=1}^{r} A_{i-1} f(x+[i-(r+1)]t) = O(t^k) \quad \text{for almost every } x \in E. \tag{1.2}$$

Integrating (1.1) and (1.2), we obtain

$$\sum_{i=1}^{r} \frac{A_{i-1}}{i} D^{-1}f(x+it) - \left[\sum_{i=1}^{r} \frac{A_{i-1}}{i}\right] D^{-1}f(x) = O(t^{k+1}) \tag{1.3}$$

for almost every $x \in E$; and

$$\sum_{j=1}^{r} \frac{A_{j-1}}{j-(r+1)} D^{-1}f(x+[j-(r+1)]t) - \left[\sum_{j=1}^{r} \frac{A_{j-1}}{j-(r+1)}\right] D^{-1}f(x) = O(t^{k+1}) \tag{1.4}$$

for almost every $x \in E$. Sliding equation (1.4) to the right by r and changing indices by setting $i=j-1$, we have

$$\sum_{i=0}^{r-1} \frac{A_i}{i-r} D^{-1}f(x+it) - \left[\sum_{i=0}^{r-1} \frac{A_i}{i-r}\right] D^{-1}f(x+rt) = O(t^{k+1}) \tag{1.5}$$

for almost every $x \in E$.

We now show that the coefficients of (1.3) and (1.5) are not proportional. If the derivatives in (1.3) and (1.5) have been suitably normalized (see footnote ([3])), then when they are tested on the function $g(x)=x^{k+1}/(k+1)!$, both are identically equal to t^{k+1}. Hence if the coefficients of the derivatives in (1.3) and (1.5) are proportional, they must be equal. Suppose this is the case. Equating the coefficients of $f(x+it)$, $i=1, 2, \ldots, r-1$, we have

$$\frac{A_{i-1}}{i} = \frac{A_i}{i-r}, \qquad i = 1, \ldots, r-1,$$

which yields recursively,

$$\begin{aligned}
A_1 &= \frac{1-r}{1} A_0 = (-1)\binom{r-1}{1} A_0, \\
A_2 &= \frac{2-r}{2}\cdot\frac{1-r}{1} A_0 = (-1)^2\binom{r-1}{2} A_0, \\
&\vdots \\
A_{r-1} &= \frac{(r-1)-r}{(r-1)}\cdot\frac{(r-2)-r}{r-2}\cdots\frac{1-r}{1} A_0 = (-1)^{r-1}\binom{r-1}{r-1} A_0.
\end{aligned}$$

But $r-1=k+l>k$, so

$$\sum_{i=1}^{r} A_{i-1} i^k = A_0 \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} i^k = 0$$

contrary to the assumption that (1.1) is a kth generalized derivative.

Hence the derivatives in (1.3) and (1.5) do not have all coefficients equal. Therefore there is an $i_0 \in \{0, 1, \ldots, r\}$ such that the coefficients of $f(x+i_0 t)$, call them

a and b, are unequal. Set $\alpha=b(b-a)^{-1}$ and $\beta=-a(b-a)^{-1}$ and consider the derivative formed by adding the derivative in (1.3) multiplied by α to the derivative in (1.5) multiplied by β. Since $\alpha+\beta=1$, this is a $(k+1)$th generalized derivative which is properly normalized, as can be seen by testing it on g. Since $\alpha a+\beta b=0$, the coefficient of $f(x+i_0t)$ is equal to zero, so that this derivative is based on the set $B=\{1, 2, \ldots, i_0-1, i_0+1, \ldots, k+l+1\}$ and hence has excess $\leqq l-1$. Then from (1.3) and (1.5) we note that $D^{-1}f\in G_{k+1}(x, B)$ for almost every $x\in E$.

Finally, let the a_i's be arbitrary. Lemma 4 will be proved if we can show that there is a $C=\{C_0, \ldots, C_{k+l+s_2}; c_0, \ldots, c_{k+l+s_2}\}$ such that all the c_i's are integers and $D^{-s_2}f\in G_{k+s_2}(x, C)$ for almost every $x\in E$. Our hypothesis is that

$$(1.6)\qquad \sum_{i=0}^{k+l} A_i f(x+a_i t) = O(t^k) \qquad \text{for all } x\in E.$$

Let $M\subset\{a_0, \ldots, a_{k+l}\}$ be a commensurable set, i.e., there is a real number q such that mq is an integer for every $m\in M$. Let M be of maximal cardinality, i.e., if $N\subset\{a_0, \ldots, a_{k+l}\}$ is a commensurable set, then N has no more elements than does M. Replacing t by qt, we may assume without loss of generality, that all the elements of M are integers. By the sliding lemma we may assume that no $a_i=0$. Let n be any integer $\notin\{0, a_0, \ldots, a_{k+l}\}$. Integrating equation (1.6) we have

$$(1.7)\qquad \sum_{i=0}^{k+l} \frac{A_i}{a_i} D^{-1}f(x+a_i t) - \left[\sum_{i=0}^{k+l} \frac{A_i}{a_i}\right] D^{-1}f(x) = O(t^{k+1})$$

for all $x\in E$. From equation (1.6) and the sliding lemma, we have

$$\sum_{i=0}^{k+l} A_i f(x+[a_i-n]t) = O(t^k)$$

for almost every $x\in E$. Integrating this and then sliding the result back to the right by n, we have

$$(1.8)\qquad \sum_{i=0}^{k+l} \frac{A_i}{a_i-n} D^{-1}f(x+a_i t) - \left[\sum_{i=0}^{k+l} \frac{A_i}{a_i-n}\right] D^{-1}f(x+nt) = O(t^{k+1})$$

for almost every $x\in E$.

If $M=\{a_0, \ldots, a_{k+l}\}$, the conclusion is immediate with $s_2=0$, $c_i=a_i$. If not, we pick $a_j\in\{a_0, \ldots, a_{k+l}\}$ such that $a_j\notin M$. Since $n\neq 0$, we may set $\gamma=a_j/n$ and $\delta=(n-a_j)/n$. We find that $\gamma+\delta=1$ and $(A_j/a_j)\gamma+(A_j/(a_j-n))\delta=0$. Therefore if we add the derivative in (1.7) multiplied by γ to the derivative in (1.8) multiplied by δ, by the argument preceding (1.6), the resultant $(k+1)$th derivative is normalized, and has zero for the coefficient of $D^{-1}f(x+a_jt)$. Furthermore $D^{-1}f$ is $(k+1)$ generalized-bounded with respect to this derivative at almost every $x\in E$. If

$$\{a_0, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{k+l}, 0, n\}$$

is a commensurable set, the conclusion has been reached with $s_2=1$. If not, we pick $a_j'\in\{a_0, \ldots, a_{k+l}\}$ such that $a_j'\notin M\cup\{a_j\}$ and repeat the argument. At least one element of the set of a_i's that are not members of M is removed at each step, so we obtain the desired result with $s_2 \leqq$ the cardinality of this set.

Proof of Theorem 1. From Lemma 4 we have that for some integer s, $D^{-s}f$ is $k+s$ Riemann-bounded for almost every $x \in E$. From Theorem 1 of [2] it follows that $D^{-s}f \in t_{k+s}(x)$ for almost every $x \in E$.

Since $D^{-s}f \in t_{k+s}(x)$ for almost every $x \in E$, there is a perfect set $\Pi \subset E$ of measure arbitrarily close to E and there are functions F_1 and F_2 such that

(1) $D^{-s}f = F_1 + F_2$,

(2) F_1 has $k+s$ ordinary continuous derivatives on E,

(3) $F_2 = 0$ on Π.

See [4] for this decomposition. Define D^i to be the ordinary ith derivative. From (2) it follows that if we set $f_1 = D^s F_1$, then f_1 has k continuous derivatives on E. Almost everywhere on E, $D^s D^{-s} f$ exists and is equal to f. Hence

$$D^s F_2 = D^s[D^{-s}f - F_1]$$

exists and is equal to $f - f_1$ almost everywhere on E. Set $f_2 = D^s F_2$. Since Π is perfect,

$$D^1F_2(x) = \lim_{t \to 0} \frac{F_2(x+t) - F_2(x)}{t} = \lim_{x+t_n \in \Pi; t_n \to 0} \frac{F_2(x+t_n) - F_2(x)}{t_n} = 0,$$

$$D^2F_2(x) = D^1[D^1F_2(x)] = 0,$$

and $\cdots$

$$D^sF_2(x) = D[D^{s-1}F_2(x)] = 0$$

almost everywhere on Π. Since f and $f_1 \in G_k(x, A)$ for almost every $x \in E$, $f_2 \in G_k(x, A)$ for almost every $x \in E$. It suffices to prove that $f_2 \in t_k(x)$ for almost every $x \in \Pi$. Lemma 7 of [2] states that if a function belongs to $T_k(x)$ on a set, then almost everywhere on that set the function belongs to $t_k(x)$. Hence Theorem 1 is proved if we can show that $f_2(x+t) = O(t^k)$ when x is a point of density of Π. We may assume that

$$\left| \sum_{i=0}^{k+l} A_i f_2(x + a_i t) \right| \leqq M|t|^k \quad \text{if} \quad |t| < \delta \text{ for all } x \in \Pi.$$

Let 0 be a point of density of Π. By Lemma 1, if $t > 0$ is sufficiently small, there is a $u \in [t, 2t]$ such that all of the points

$$t - a_0 u,$$
$$t + (a_i - a_0)u, \qquad i = 1, \ldots, k+l,$$

belong to Π (if $a_0 = 0$, reorder the first two terms). Then

$$f_2(t + (a_i - a_0)u) = 0, \qquad i = 1, 2, \ldots, k+l,$$

so that

$$|A_0 f_2(t)| = \left| \sum_{i=0}^{k+l} A_i f_2([t - a_0 u] + a_i u) \right| \leqq Mu^k \leqq 2^k M t^k$$

if t is sufficiently small. This shows that $f_2 \in T_k(0)$ and completes the proof of Theorem 1.

REMARK. We may strengthen Theorem 1 by weakening the hypothesis to

$$\sum_{i=0}^{k+l} A_i f(x+a_i t) = O(t^k) \qquad \text{as } t \to +0 \text{ for all } x \in E$$

while still obtaining the same conclusion that $f \in t_k(x)$ for almost every $x \in E$.

The proof of Theorem 1 may be followed line for line with the following exceptions. We must invoke Theorem 7 of [2] instead of Theorem 1 of that paper. Theorem 7 states that if f is k Riemann-bounded *as* $t \to +0$ at each x of E, then $f \in t_k(x)$ for almost every $x \in E$. Also the reference to Lemma 7 of [2] must be replaced by one to Theorem 8 of that paper which states that if f is Peano-bounded of order k *as* $t \to +0$ for all $x \in E$, then $f \in t_k(x)$ for almost every $x \in E$.

2. **An L^p generalization of the Riemann derivative.** Now let $1 \leqq p < \infty$ and let $f \in L^p[x-\varepsilon, x+\varepsilon]$ for some $\varepsilon > 0$. We may extend all of the definitions of derivatives given in §1 to definitions of derivatives in L^p.

A function f is said to have at x a kth Peano derivative in L^p, i.e., $f \in t_k^p(x)$ if there are constants $f_0(x), \ldots, f_k(x)$ such that

$$\left(\frac{1}{h}\int_0^h \left| f(x+t) - \left\{ f_0(x) + \cdots + \frac{f_k(x)}{k!} t^k \right\}\right|^p dt\right)^{1/p} = o(h^k) \qquad \text{as } h \to 0.$$

Let $A = \{A_0, \ldots, A_{k+l}; a_0, \ldots, a_{k+l}\}$ be such that

$$\sum_{i=0}^{k+l} A_i a_i^j = 0, \qquad j = 0, 1, \ldots, k-1,$$
$$= k!, \qquad j = k.$$

We say that f is k-generalized-bounded in L^p with respect to A at x, i.e., $f \in G_k^p(x, A)$, if

$$\left(\frac{1}{h}\int_0^h \left|\sum_{i=0}^{k+l} A_i f(x+a_i t)\right|^p dt\right)^{1/p} = O(h^k) \qquad \text{as } h \to 0.$$

The classes $T_k^p(x)$ and $g_k^p(x, A)$ are also defined by replacing the L^∞ norm of §1 by the L^p norm. As in §1, if $f \in G_k^p(x, A)$ in the special case when

$$A = \left\{(-1)^k \binom{k}{0}, \ldots, (-1)^0 \binom{k}{k}; -\frac{k}{2}, \ldots, \frac{k}{2}\right\},$$

f is said to be k-Riemann-bounded in L^p at x. All the relations depicted in Figure 1 are still valid if the superscript p is attached to the name of each class. Parallel to Theorem 1 we have

THEOREM 2. *If $f \in G_k^p(x, A)$ for all $x \in E$, then $f \in t_k^p(x)$ for almost every $x \in E$.*

Mary Weiss has proved that if f has a kth symmetric L^p derivative for all $x \in E$, then $f \in t_k^p(x)$ for almost every $x \in E$ [5]. Since the existence of the kth symmetric

L^p derivative at x implies the existence of the kth Riemann L^p derivative at x, her result is contained in Theorem 2.

Proof. If $g \in L^p$, by Hölder's inequality we have

$$\left|\frac{1}{h}\int_0^h g(t)\,dt\right| \leqq \frac{1}{h}\int_0^h |g(t)|\,dt \leqq \left(\frac{1}{h}\int_0^h |g(t)|^p\,dt\right)^{1/p}.$$

Hence from the hypothesis of Theorem 2 we may deduce that

$$\frac{1}{h}\int_0^h \sum_{i=0}^{k+l} A_i f(x+a_i t)\,dt = O(h^k) \qquad \text{at } h \to 0 \tag{2.1}$$

for every $x \in E$. If some $a_i=0$, say $a_0=0$, then

$$\frac{1}{2^k-1}\sum_{i=0}^{k+l} A_i[f(x+2a_i t)-f(x+a_i t)]$$

still is a kth generalized derivative, since

$$\begin{aligned}\frac{1}{2^k-1}\sum_{i=0}^{k+l} A_i a_i^j(2^j-1) &= 0, \qquad j<k,\\ &= k!, \qquad j=k.\end{aligned}$$

Further,

$$\begin{aligned}\frac{1}{h}\int_0^h \sum_{i=0}^{k+l} A_i f(x+2a_i t)\,dt &= \frac{1}{2h}\int_0^{2h}\sum_{i=0}^{k+l} A_i f(x+a_i s)\,ds\\ &= O((2h)^k) = O(h^k) \qquad \text{as } h\to 0\end{aligned}$$

for every $x \in E$. Hence we may assume that (2.1) holds with no $a_i=0$.

Multiplying (2.1) by h and performing the integration, we have

$$\sum_{i=0}^{k+l}\frac{A_i}{a_i}D^{-1}f(x+a_i h)-\sum_{i=0}^{k+l}\frac{A_i}{a_i}D^{-1}f(x) = O(h^{k+1})$$

as $h\to 0$ for every $x\in E$. Since

$$\sum_{i=0}^{k+l}\frac{A_i}{a_i}-\sum_{i=0}^{k+l}\frac{A_i}{a_i}=0$$

and

$$\begin{aligned}\sum_{i=0}^{k+l}\left(\frac{A_i}{a_i}\right)a_i^j &= 0, \qquad 1\leqq j\leqq k,\\ &= k!, \qquad j=k+1,\end{aligned}$$

after multiplication by $k+1$ we have that $D^{-1}f$ is $(k+1)$ generalized-bounded with respect to

$$\left\{-\sum_{i=0}^{k+l}\frac{A_i}{a_i},\frac{A_0}{a_0},\dots,\frac{A_{k+l}}{a_{k+l}};\ 0,a_0,\dots,a_{k+l}\right\}$$

at every $x\in E$. By Theorem 1, $D^{-1}f\in t_{k+1}(x)$ for almost every $x\in E$.

As in the proof of Theorem 1, set $D^{-1}f=G+L$ where G is $k+1$ times continuously differentiable throughout the domain of $D^{-1}f$ and where $L=0$ on an arbitrarily large perfect subset Π of E. Then on E (assuming we have removed the points where $D^1D^{-1}f\neq f$), $D^{-1}f$ and G are both differentiable and hence so is $L=D^{-1}f-G$. In particular, L is differentiable on Π and since Π is perfect, $D^1L=l=0$ on Π. Also $f\in G_k^p(x, A)$ and $D^1G\in t_k(x)\subseteq G_k^p(x, A)$ if $x\in\Pi\subseteq E$, so that $l\in G_k^p(x, A)$ for every point of Π.

It suffices to prove that l, and hence f, belongs to $t_k^p(x)$ for almost every $x\in\Pi$. To do this, it suffices to prove that $l\in T_k^p(x)$ and that $l_i(x)=0$, $i=0, 1, \ldots, k-1$, at each point of density of Π. For from this result, by Theorem 10 of [6], it follows that $l\in t_k^p(x)$ at almost every x in Π. We collect what remains to be proved of Theorem 2 into a lemma.

LEMMA 5. *Let $l(x)=0$ on E, $|E|>0$, $l\in L^p$, and*

$$\int_0^h\left|\sum_{i=0}^n A_i l(x+a_it)\right|^p dt = O(h^\alpha) \qquad \text{as } h\to 0$$

for all $x\in E$, where $\alpha>1$, $p\geqq 1$ (we actually only need the case of $\alpha=kp+1$). Then

$$\int_{-h}^h |l(x+t)|^p\, dt = O(h^\alpha) \qquad \text{as } h\to 0$$

for almost every $x\in E$.

(The proof follows that of a similar lemma in [5].)

Proof. As in the proof of Lemma 2, without loss of generality we may assume that

$$\int_0^h\left|\sum_{i=0}^n A_i l(x+a_it)\right|^p dt \leqq M|h|^\alpha \qquad \text{if } |h|<\delta \tag{2.2}$$

for all $x\in E$. It suffices to prove this lemma for each point of density of E. Let $x=0$ be such a point. We must show

$$\int_{-h}^h |l(t)|^p\, dt = O(h^\alpha) \qquad \text{as } h\to 0. \tag{2.3}$$

Assume that $a_0\neq 0$ in (2.2) (if it does, reorder). If $a_0\neq 1$, we set $s=a_0t$, and divide by $|A_0|$:

$$\left|\int_0^{a_0h}\left|l(x+s)+\sum_{i=1}^n A_0^{-1}A_i l\left(x+s\frac{a_i}{a_0}\right)\right|^p ds\right| \leqq \frac{M}{|A_0a_0^{\alpha-1}|}|a_0h|^\alpha$$

if $|h|<\delta$, i.e., if $|a_0h|<|a_0|\delta$. Hence we may assume that $a_0=A_0=1$ in (2.2).

Either

$$\int_0^h |l(t)|^p\, dt \geqq \frac{1}{2}\int_{-h}^h |l(t)|^p\, dt \tag{2.4}$$

or

$$\int_0^h |l(t)|^p\, dt < \frac{1}{2}\int_{-h}^h |l(t)|^p\, dt.$$

Suppose that the former holds (the argument is essentially the same if the latter holds). For definiteness, assume $h>0$.

Let F be the complement of E. Pick $x \in [-h/2, 0] \cap E$ (this can be done if h is sufficiently small since 0 is a point of density).

$$
\begin{aligned}
\int_0^h |l(t)|^p\,dt &\leqq \int_0^{h-x} |l(x+t)|^p\,dt \\
&\leqq \int_0^{h-x} \left| l(x+t)+\sum_{i=1}^n A_i l(x+a_i t)\right|^p dt + \int_{A(x,h)} |l(x+t)|^p\,dt
\end{aligned}
\tag{2.5}
$$

where $A(x, h)=\{t \in [0, h-x] \mid (x+a_i t) \in F \text{ for some } i \in N=\{1,\ldots,n\}\}$.

$$
\int_{A(x,h)} |l(x+t)|^p\,dt \leqq \sum_{i=1}^n \int_{A_i(x,h)} |l(t)|^p\,dt, \tag{2.6}
$$

since

$$
A(x, h) \subseteq \bigcup_{i=1}^n A_i(x, h)
$$

where $A_i(x, h)=\{s \in [0, h-x] \mid x+a_i s \in F\}=\{t \in [x, h] \mid x(1-a_i)+a_i t \in F\}$ (the last equality coming from the substitution $s=x+t$).

Suppose that we can prove:

(2.7) There is an $x \in [-h/2, 0] \cap E$ such that

$$
\int_{A_i(x,h)} |l(t)|^p\,dt \leqq \frac{1}{2n}\int_0^h |l(t)|^p\,dt, \qquad i = 1, 2, \ldots, n.
$$

Then from (2.2), (2.5), (2.6), and (2.7) we will have

$$
\frac{1}{2}\int_0^h |l(t)|^p\,dt \leqq M(h-x)^\alpha \leqq M(2h)^\alpha = 2^\alpha M h^\alpha
$$

which when combined with (2.4), yields

$$
\int_{-h}^h |l(t)|^p\,dt \leqq 2\int_0^h |l(t)|^p\,dt \leqq 2^{\alpha+2} M h^\alpha
$$

which in turn implies (2.3), as required.

Only (2.7) remains to be proved. Set $M=\max_{i\in N}\{1/2+(3/2)|a_i|\}$. Define $\varepsilon=\varepsilon(h)$ by

$$
\varepsilon = \frac{1}{h}\int_{-Mh}^{Mh} \chi_F(v)\,dv
$$

where χ_F is the characteristic function of F. Note that 0 is a point of rarefaction of F, so ε can be made arbitrarily small by choosing h sufficiently small. Set

$$
I_i(x) = \int_{A_i(x,h)} |l(t)|^p\,dt = \int_x^h |l(t)|^p \chi_F(x[1-a_i]+a_i t)\,dt.
$$

Then

$$\int_{-h/2}^{0} I_i(x)\,dx \leqq \int_{-h/2}^{0}\left[\int_{-h}^{h} |l(t)|^p \chi_F(x[1-a_i]+a_i t)\,dt\right] dx$$

$$\leqq \frac{1}{|1-a_i|}\int_{-Mh}^{Mh}\int_{-h}^{h} |l(u)|^p \chi_F(v)\,du\,dv,$$

setting $u=t$, $v=a_i t+(1-a_i)x$ and noting $|(\partial(t, x)/\partial(u, v))| = (1/|1-a_i|)$. Hence

$$\int_{-h/2}^{0} I_i(x)\,dx \leqq \frac{1}{|1-a_i|}\left[\int_{-Mh}^{Mh} \chi_F(v)\,dv\right]\left[\int_{-h}^{h} |l(t)|^p\,dt\right]$$

using Fubini's theorem and its converse freely since all functions are positive and integrable. By (2.4) and the definition of ε,

$$\int_{-h/2}^{0} I_i(x)\,dx \leqq \frac{1}{|1-a_i|}\,\varepsilon h\cdot 2\int_{0}^{h} |l(t)|^p\,dt.$$

By Tchebycheff's inequality, if

$$B_i = \left\{x \in \left[-\frac{h}{2}, 0\right] \cap E \,|\, I_i(x) > \frac{1}{2n}\int_0^h |l(t)|^p\,dt\right\},$$

then

$$|B_i|\cdot\frac{1}{2n}\int_0^h |l(t)|^p\,dt \leqq \frac{2\varepsilon h}{|1-a_i|}\int_0^h |l(t)|^p\,dt,$$

i.e.,

$$|B_i| \leqq \frac{4n\varepsilon h}{|1-a_i|}.$$

Picking h so small that

$$\varepsilon < \frac{\min_{i\in N} |1-a_i|}{16n^2},$$

we have

$$|B_i| < \frac{4nh}{|1-a_i|}\cdot\frac{\min_{i\in N}|1-a_i|}{16n^2} \leqq \frac{h}{4n}$$

so that

$$\left|\bigcup_{i=1}^{n} B_i\right| \leqq \sum_{i=1}^{n} |B_i| < \frac{h}{4}.$$

If we further choose h so small that $|[-h/2, 0] \cap E| \geqq h/4$, we can find an x such that

$$x \in \left[-\frac{h}{2}, 0\right] \cap E \quad\text{and}\quad x \notin \bigcup_{i=1}^{n} B_i.$$

This is the x required for (2.7). This completes the proof of Theorem 2.

REMARK. As in §1, we may weaken the hypothesis of Theorem 2 to

$$\left(\frac{1}{h}\int_0^h \left|\sum_{i=0}^{k+l} A_i f(x+a_i t)\right|^p dt\right)^{1/p} = O(h^k) \qquad \text{as } h \to +0$$

for all $x \in E$, while still obtaining the conclusion $f \in t_k^p(x)$ for almost every $x \in E$.

The proof of this remark follows the proof of Theorem 2 except that Theorem 1 is replaced by the remark at the end of §1, Theorem 10 of [6] is coupled with Lemma 6 below, and Lemma 5 is replaced by Lemma 7 below.

LEMMA 6. *Let $\alpha > 0$. If*

$$\int_0^h |l(x+t)|^p\, dt = O(h^\alpha) \qquad \text{as } h \to +0 \text{ for all } x \in E,$$

then

$$\int_{-h}^h |l(x+t)|^p\, dt = O(h^\alpha) \qquad \text{as } h \to +0 \text{ for almost every } x \in E.$$

The same conclusion holds if our hypothesis is

$$\int_{-h}^0 |l(x+t)|^p\, dt = O(h^\alpha) \text{ as } h \to +0 \qquad \text{for all } x \in E.$$

Proof. We prove only the former statement, since the proof of the latter is similar. As in Lemma 5, without loss of generality we may assume

$$\int_0^h |l(x+t)|^p\, dt \leqq Mh^\alpha \qquad \text{if } 0 < h < \delta \text{ for all } x \in E.$$

To simplify notation, let 0 be a point of density of E. It suffices to show

$$\int_{-h}^h |l(t)|^p\, dt \leqq (4^\alpha M)h^\alpha$$

if h is sufficiently small. Pick $h < \delta/4$ so small that $[-2h, -h] \cap E \neq \varnothing$. Pick $-k \in [-2h, -h] \cap E$. Then $0 < k < \delta/2$, $2k < 4h < \delta$, and

$$\begin{aligned}\int_{-h}^h |l(t)|^p\, dt \leqq \int_{-k}^k |l(t)|^p\, dt &= \int_0^{2k} |l(-k+t)|^p\, dt \\ &\leqq M(2k)^\alpha \leqq M(4h)^\alpha = (4^\alpha M)h^\alpha.\end{aligned}$$

LEMMA 7. *If $l(x)=0$ on E, $|E| > 0$, $l \in L^p$ for all $x \in E$*

$$\int_0^h \left|\sum_{i=0}^n A_i l(x+a_i t)\right|^p dt = O(h^\alpha) \qquad \text{as } h \to +0$$

where $\alpha > 1$, $p \geqq 1$, then

$$\int_{-h}^h |l(x+t)|^p\, dt = O(h^\alpha) \qquad \text{as } h \to +0 \text{ for almost every } x \in E.$$

Proof. Because of Lemma 6, it suffices to show

$$\int_0^h |l(x+t)|^p\, dt = O(h^\alpha) \qquad \text{as } h \to +0$$

for almost every $x \in E$. Assume that $a_0 > 0$. As in the proof of Lemma 5, we may then assume $A_0 = a_0 = 1$.

(If $a_0 < 0$, the hypothesis can be reduced to

$$\int_{-h}^0 \left| l(x+t) + \sum_{i=1}^n A_i l(x+a_i t) \right|^p dt = O(h^\alpha) \qquad \text{as } h \to +0 \text{ for all } x \in E.$$

The proof then proceeds to the conclusion that

$$\int_{-h}^0 |l(x+t)|^p\, dt = O(h^\alpha) \qquad \text{as } h \to +0 \text{ for almost every } x \in E.$$

Now apply the second part of Lemma 6 to produce the desired conclusion.)

As in Lemma 5, without loss of generality we may assume

$$\int_0^h \left| l(x+t) + \sum_{i=1}^n A_i l(x+a_i t) \right|^p dt \leqq Mh^\alpha, \qquad 0 < h < \delta \text{ for all } x \in E.$$

By discarding a subset of measure zero we may further assume that every point of E is a point of differentiability for $\int_0^x |l(t)|^p\, dt$, so that for all $x \in E$,

$$D\left(\int_0^x |l(t)|^p\, dt\right) = |l(x)|^p = 0.$$

Let 0 be a point of density of E. It suffices to show

$$\int_0^h |l(t)|^p\, dt = O(h^\alpha) \qquad \text{as } h \to +0.$$

Denote $\int_a^b |l(t)|^p\, dt$ by $I(a, b)$. We divide the proof into three cases.

Case I. $I(h/2, h) > I(0, h)/2$.

In this case the proof is very much like that of Lemma 5. Pick $x \in [0, h/2] \cap E$. As in Lemma 5,

$$\text{(2.8)} \qquad \begin{aligned} I\left(\frac{h}{2}, h\right) &= \int_{h/2}^h |l(t)|^p\, dt \leqq \int_x^h |l(t)|^p\, dt \\ &\leqq \int_0^{h-x} \left| l(x+t) + \sum_{i=1}^n A_i l(x+a_i t) \right|^p dt + \sum_{i=1}^n \int_{A_i(x,h)} |l(t)|^p\, dt \end{aligned}$$

where $A_i(x, h) = \{t \in [x, h] \mid [1-a_i]x + a_i t \notin E\}$. As in Lemma 5, if h is sufficiently small, an $x \in [0, h/2] \cap E$ may be found such that for all $i = 1, 2, \ldots, n$

$$\int_{A_i(x,h)} |l(t)|^p\, dt \leqq \frac{1}{4n} I(0, h) < \frac{1}{2n} I\left(\frac{h}{2}, h\right).$$

For this x, (2.8) implies

$$I(0,h) < 2I\left(\frac{h}{2},h\right) \leqq 4\int_0^{h-x} \left|l(x+t)+\sum_{i=1}^{n} A_i l(x+a_i t)\right|^p dt \leqq 4M(h-x)^\alpha \leqq 4Mh^\alpha$$

if h is sufficiently small.

Case II.

$$I\left(\frac{h}{2^{i+1}},\frac{h}{2^i}\right) \leqq \frac{1}{2}I\left(0,\frac{h}{2^i}\right), \qquad i = 0,1,\dots,k-1,$$

but

$$I\left(\frac{h}{2^{k+1}},\frac{h}{2^k}\right) > \frac{1}{2}I\left(0,\frac{h}{2^k}\right). \tag{2.9}$$

In this case we have

$$I\left(0,\frac{h}{2^i}\right) = I\left(0,\frac{h}{2^{i+1}}\right)+I\left(\frac{h}{2^{i+1}},\frac{h}{2^i}\right) \qquad \text{for all } i,$$

so

$$I\left(0,\frac{h}{2^i}\right) \leqq I\left(0,\frac{h}{2^{i+1}}\right)+\frac{1}{2}I\left(0,\frac{h}{2^i}\right), \qquad i = 0,1,\dots,k-1.$$

Hence

$$I\left(0,\frac{h}{2^i}\right) \leqq 2I\left(0,\frac{h}{2^{i+1}}\right), \qquad i = 0,1,\dots,k-1,$$

so that

$$I(0,h) \leqq 2I\left(0,\frac{h}{2}\right) \leqq 2^2I\left(0,\frac{h}{2^2}\right) \leqq \cdots \leqq 2^kI\left(0,\frac{h}{2^k}\right). \tag{2.10}$$

If h is sufficiently small, applying Case I to (2.9), we have

$$I\left(0,\frac{h}{2^k}\right) < 4M\left(\frac{h}{2^k}\right)^\alpha.$$

Combining this with (2.10), we have

$$I(0,h) < 2^k\left[4M\left(\frac{h}{2^k}\right)^\alpha\right] < 4Mh^\alpha \qquad \text{since } \alpha > 1.$$

Case III. $I(h/2^{i+1}, h/2^i) \leqq (1/2)I(0, h/2^i)$ for all $i=0, 1, 2, \dots$.

Reasoning as in Case II, we arrive at (2.10) for every k. Dividing through by h, we obtain

$$\frac{1}{h}I(0,h) \leqq \frac{1}{(h/2^k)}I\left(0,\frac{h}{2^k}\right), \qquad k = 1,2,\dots.$$

Since by assumption, $0 \in E$ is a point of differentiability of the integral, the right-hand side tends to 0 as k tends to infinity. Hence

$$\frac{1}{h} I(0, h) = 0, \qquad I(0, h) = 0,$$

which is surely $O(h^\alpha)$. Q.E.D.

3. A generalization of smoothness.

THEOREM 3. *Let a, b, c be any distinct real numbers, and let A, B, C be real numbers such that $A+B+C=0$, and not all of A, B, $C=0$. Suppose that for all $x \in E$,*

$$Af(x+at)+Bf(x+bt)+Cf(x+ct) = O(t) \qquad \text{as } t \to 0.$$

Then:

(a) *if $Aa+Bb+Cc \neq 0$, $f \in t_1(x)$ for almost every $x \in E$.*

(b) *if $Aa+Bb+Cc=0$, $f \in \Lambda_1(x)$*, i.e.,

$$f(x+t)+f(x-t)-2f(x) = O(t) \qquad \text{as } t \to 0$$

for almost every $x \in E$.

(c) *if the "O" in the hypothesis of* (b) *is replaced by "o", it may also be replaced by "o" in the conclusion, i.e., f is then smooth almost everywhere in E.*

Proof. Part (a) was first proved by J. P. Kahane. It can be rephrased: if $f \in G_1(x, \{A, B, C; a, b, c\})$ for all $x \in E$, then $f \in t_1(x)$ for almost every $x \in E$. Thus stated, it is seen to be a special case of Theorem 1.

To prove (b), assume, for example, that $A \neq 0$. By the sliding lemma, $Af(x+[a-c]t)+Bf(x+[b-c]t)+Cf(x)=O(t)$ for almost every $x \in E$. Set $h=(b-c)t$, $\alpha=(a-c)/(b-c)$:

$$Af(x+\alpha h)+Bf(x+h)+Cf(x) = O(h) \tag{3.1}$$

for almost every $x \in E$.

$$\begin{aligned} A\alpha+B &= \frac{1}{b-c}[A(a-c)+B(b-c)] = \frac{1}{b-c}[aA+bB-c(A+B)] \\ &= \frac{1}{b-c}[aA+bB+cC] \qquad (\text{since } A+B+C=0) \\ &= 0 \qquad\qquad\qquad\qquad\quad (\text{since } Aa+Bb+Cc=0). \end{aligned}$$

Hence $B=-\alpha A$, $C=-(A+B)=(\alpha-1)A$ so that (3.1) becomes after division by A

$$f(x+\alpha h)-\alpha f(x+h)+(\alpha-1)f(x) = O(h) \tag{3.2}$$

for almost every $x \in E$. Slide this result by $-\alpha$:

$$f(x)-\alpha f(x+[1-\alpha]h)+(\alpha-1)f(x-\alpha h) = O(h)$$

for almost every $x \in E$. Replace h by $-h$:

$$f(x)-\alpha f(x+[\alpha-1]h)+(\alpha-1)f(x+\alpha h) = O(h)$$

for almost every $x \in E$. Add the product of equation (3.2) by $(1-\alpha)$ to this result:

$$-\alpha f(x+[\alpha-1]h)+(1-\alpha)(-\alpha)f(x+h)-\alpha(\alpha-2)f(x) = O(h)$$

for almost every $x \in E$. Divide(3) by $-\alpha$ and slide this by 1:

$$f(x+\alpha h)+(1-\alpha)f(x+2h)+(\alpha-2)f(x+h) = O(h)$$

for almost every $x \in E$. Subtract (3.2) from this:

$$(1-\alpha)f(x+2h)-2(1-\alpha)f(x+h)+(1-\alpha)f(x) = O(h)$$

for almost every $x \in E$. Slide this by -1 and divide by $(1-\alpha)$:

$$f(x+h)-2f(x)+f(x-h) = O(h)$$

for almost every $x \in E$.

To prove part (c), simply replace "O" by "o" throughout the proof of (b).

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(3) Since a, b, c are distinct, $\alpha=(a-c)/(b-c)\neq 0$ or 1, so that α and $1-\alpha$ are both not 0.