CLASSROOM CAPSULES

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Classroom Capsules are short (1–3 page) notes that contain new mathematical insights on a topic from undergraduate mathematics, preferably something that can be directly introduced into a college classroom as an effective teaching strategy or tool. Classroom Capsules should be prepared according to the guidelines on the inside front cover and sent to any of the above editors.

Series Involving Iterated Logarithms
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Almost every standard calculus text shows that $\sum \frac{1}{n}$ diverges but that when $p > 1$, $\sum \frac{1}{n^p}$ converges. This is often called the $p$-test. The $p$-test always interested us, since we felt that there was not much difference between, say, $\sum \frac{1}{n}$ and $\sum \frac{1}{n^{1.001}}$. Some texts, for example [4], go on to interpolate closer divergent/convergent pairs of series: for $P$ greater than 1, the pair $\left( \sum \frac{1}{n^{\ln n}}, \sum \frac{1}{n^{(\ln n)^p}} \right)$ are closer to each other and, for $Q > 1$, the pair $\left( \sum \frac{1}{n \ln n \ln \ln n}, \sum \frac{1}{n \ln n \ln \ln n \ln \ln \ln n} \right)$ are still closer. By “closer” we mean that $\frac{1}{n} \gg \frac{1}{n \ln n} \gg \frac{1}{n \ln n \ln n} \gg \frac{1}{n \ln n \ln n \ln \ln n} \gg \frac{1}{n \ln n \ln n \ln \ln n \ln \ln \ln n}$, where $b_n \gg c_n$ means that $\lim_{n \to \infty} \frac{b_n}{c_n} = \infty$. Furthermore these relations hold even if $p$ is very close to 1 while $P$ (or $Q$) is very large.

Let $b > 1$ be any base. Denote the iteration of $k$ logarithms, $\log_b \log_b \cdots \log_b x$, by $\log_x^k x$; in particular when $b = e$, we write $\ln \ln \cdots \ln x = \ln^k x$.

**Theorem 1.** Let $M = M(b, k)$ be so large that $\log_x^k M > 0$. Then

$$S = S(b, k, p) = \sum_{n \geq M} \frac{1}{n \log_b n \log_b^2 n \cdots \log_b^{k-1} n (\log_b^k n)^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Notice that the $p$-test is the case $S(e, 0, p)$, if we interpret $\ln^0 n$ to be $n$, while the assertion that the other two pairs mentioned above are divergent/convergent pairs are the cases $S(e, 1, p)$ and $S(e, 2, p)$. The well known Integral Test proof of $S(e, 1, p)$ easily extends to a proof of $S(e, k, p)$ for every $k$. There is a less well known, integration-avoiding, proof for $S(2, k, p)$. We will extend both proofs, thereby giving two proofs of Theorem 1.

As we mentioned above, as $k$ increases, the $k$th cases of Theorem 1 appear to provide better and better demarcation of a diverge/converge line. This was the basis for our interest in the $b = e$ cases of Theorem 1, when we were learning calculus. This led us to hope that there might be a series that is right on the diverge/converge line.
Unfortunately, there does not exist a smallest divergent series nor a largest convergent one [1]. In a way, this says that in a quest to locate a diverge/converge line, we cannot hope to do much better than something like Theorem 1.

**Lemma 1.** Let \( r := \log_b c \). Then for each positive integer \( k \),

\[
\lim_{x \to \infty} \frac{\log_b^k x}{\log_x^k x} = r.
\]

Lemma 1 is a not so obvious generalization of the standard logarithmic identity \( \frac{\log_b x}{\log_x x} = r \). It will facilitate our proof of Theorem 1.

To prove the lemma, set \( L(x) := \log_x x \), so that \( \log_b x = \log_b c \log_x x = rL(x) \). Using the convention introduced above for functional iteration, we need to prove

\[
\lim_{x \to \infty} \frac{(rL)^k x}{L^k x} = r.
\]

But \( (rL)^k x = r(Lr)^{k-1} Lx \), so this is equivalent to

\[
\lim_{x \to \infty} \frac{(Lr)^{k-1} Lx}{L^k x} = 1.
\]

This is true if \( k = 1 \). Continuing by induction

\[
(Lr)^k Lx = L \left[ r \left( (Lr)^{k-1} Lx \right) \right]
\]

\[
= Lr + L \left( (Lr)^{k-1} Lx \right)
\]

\[
= Lr + L \left( L^k x \left( 1 + \delta(x) \right) \right)
\]

\[
= Lr + L^{k+1} x + L \left( 1 + \delta(x) \right)
\]

\[
= L^{k+1} x + \left( Lr + \epsilon(x) \right).
\]

The second and fourth steps are due to \( L \) having the multiplicative property. The existence of a function \( \delta(x) \) which tends to 0 at \( \infty \) so that the third step holds is a reformulation of the induction hypothesis. Since \( L(1) = 0 \) and \( L \) is continuous at 1, there is a function \( \epsilon(x) \) which tends to 0 at \( \infty \) so that the last step holds. Complete the proof by dividing by \( L^{k+1} x \) and observing that \( \frac{Lr + \epsilon(x)}{L^{k+1} x} \to 0 \) as \( x \to \infty \) since the numerator is bounded while the denominator becomes infinite.

**Proof of Theorem 1.** By Lemma 1 and the Limit Comparison Test, it suffices to prove Theorem 1 for any particular choice of base. The Integral Test proof works best for base \( e \). By the Integral Test, it suffices to evaluate

\[
\int_M^\infty \frac{dx}{x \ln x \cdots (\ln^k x)^p}.
\]

This is done by means of the substitution \( u = \ln^k x \).

Our second proof of Theorem 1 avoids the use of the integral calculus. It works best for base 2 and uses the Cauchy Condensation Test. This test says that if \( a_n \searrow 0 \), then
\[ \sum a_n \text{ and } \sum 2^m a_{2^m} \text{ converge or diverge together; just group the terms of } \sum a_n \text{ into dyadic blocks to prove it [2]. For our second proof: we let } \]

\[ a_n = \frac{1}{n \log_2 n \cdots \log_2^{k-1} n (\log_2^k n)^p}. \]

Then \[ \log_2 2^n = \log_2^{j-1} (\log_2 (2^m)) = \log_2^{j-1} m \text{ and } \]

\[ 2^m a_{2^m} = \frac{1}{2^m m \cdots \log_2^{k-2} m (\log_2^{k-1} m)^p}, \]

so since the terms of \( S \) are eventually decreasing, the \( k \) case is equivalent to the \( k - 1 \) case, is equivalent to the \( k - 2 \) case, \ldots, is equivalent to the \( k = 0 \) case. Thus \( S(2, k, p) \) converges if and only if \( \sum n^{-p} \) converges. A final application of the Cauchy Condensation Test reduces this to the easy analysis of the geometric series \( \sum (2^{1-p})^n \).

\[ \blacksquare \]

**Some history.** The treatment of the natural logarithm cases \( S(e, k, p) \) by means of the Integral Test is very well known. For example, they are treated in the classic encyclopedic work of Konrad Knopp, where 4 distinct proofs are given, one of which uses Cauchy’s Condensation Test. That particular proof can easily be modified to do the cases \( S(b, k, p) \) when \( b \geq 2 \) but not the cases when \( 1 < b < 2 \) [3].

**References**


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**Sums of Integer Powers via the Stolz-Cesàro Theorem**

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One of the powerful rules for evaluating limits of sequences or series is the Stolz-Cesàro theorem which is a discrete form of l'Hôpital’s rule.

**Stolz-Cesàro Theorem.** Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of real numbers. If \( b_n \) is positive, strictly increasing and unbounded and the following limit exists:

\[ \lim_{n \to \infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = l, \]

then the limit

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = l \]

also exists and it is equal to \( l \).