Counterexample without cases

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Abstract. The standard examples of functions without a one-sided limit are defined piecewise, with at least two pieces. We give an example of a function defined everywhere by a single formula. This example is an infinite sum of simple rational functions.

Does there exist a function which does not have a right hand limit at \( x = 0 \)? The standard example is

\[
    f(x) = \begin{cases} 
    \sin \frac{1}{x} & \text{if } x \neq 0 \\
    0 & \text{if } x = 0 
    \end{cases}
\]

This example is satisfactory from almost every modern point of view, but I distinctly remember that when I first took calculus, whenever a counterexample was called for, to my slight dissatisfaction the machinery of cases was always brought in. Let me try to make this vague complaint into a precise question. What I want is an everywhere defined function given by a single formula. It would be nice if the formula could be an analytic function. But at each point of discontinuity, such a function either has a pole or an essential singularity and hence is not defined. So I am forced to relax the rule. I will allow the formula to be an infinite sum of analytic functions. It follows that, sadly, any example must be inaccessible to beginning calculus students. Notice that such a function would have been an admissible example in, say, the year 1800, while \( f(x) \) would not have been considered a function at that time. So the goal is to create an everywhere convergent infinite series of elementary functions that does not have a right hand limit at \( x = 0 \).

If \( \{s_n(x)\} \) is any sequence of analytic functions tending pointwise to \( f(x) \), then \(-s_0(x) + \sum_{k=1}^{\infty} (s_k(x) - s_{k-1}(x)) = f(x) \). This provides a general solution, but fails to satisfy an additional criterion, which is harder to make precise: that the example be as simple as possible. I will give an example which is simple enough to be accessible to students who have completed a year of calculus. If the reader can think of a simpler example, I would certainly enjoy seeing it.

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My example is an infinite sum of spikes

\[ g(x) = \sum_{n=1}^{\infty} S_{a_n, M_n}(x), \]

where a spike, defined by

\[ S_{a, M}(x) = \frac{1}{1 + M(x - a)^2}, \]

is a positive function whose maximal height 1 is attained at \( x = a \) and which is very sharply peaked if \( M > 0 \) is large. The \( \{a_n\} \) are decreasing monotonically to zero and the \( M_n \) are picked to be so large that the \( S_n(x) = S_{a_n, M_n}(x) \) satisfy

\[ \sum_{n=1}^{\infty} \max\{S_n(b_n), S_n(b_{n-1})\} < \frac{1}{2}, \]

where

\[ b_n = \frac{a_{n+1} + a_n}{2}, \quad n \geq 1, b_0 = a_1 + 1. \]

Then \( g(x) \) is large at the peaks \( \{a_k\} \),

\[ \lim_{x \to 0^+} \sup g(x) \geq \lim_{k \to \infty} \sup g(a_k) \geq \lim_{k \to \infty} S_k(a_k) = 1, \]

while \( g(x) \) is small in the valleys \( \{b_k\} \),

\[ \lim_{x \to 0^+} \inf g(x) \leq \lim_{k \to \infty} \inf \sum_{n=1}^{\infty} S_n(b_k) \]

\[ \leq \lim_{k \to \infty} \inf \sum_{n=1}^{\infty} \max\{S_n(b_n), S_n(b_{n-1})\} < \frac{1}{2}. \]

To see that \( S_n(b_k) \leq \max\{S_n(b_n), S_n(b_{n-1})\} \), note that \( S_n(b_k) \leq S_n(b_n) \) if \( k \geq n \), while \( S_n(b_k) \leq S_n(b_{n-1}) \) if \( k < n \), since \( S_n \not\uparrow \) on \((-\infty, a_n)\) and \( S_n \not\downarrow \) on \((a_n, \infty)\).

Observe that the proof that \( g(x) \) converges finitely at every \( x \) is similar to the calculation that \( g \) is small in the valleys.

In particular, if we set

\[ a_n = \frac{1}{n}, n = 1, 2, \ldots, \]

and

\[ M_n = 64n^6; \]

then for \( n \geq 1, \)

\[ S_n(b_n) = \frac{1}{1 + 64n^6 \left[ \frac{1}{\pi} \left( \frac{1}{n+1} - \frac{1}{n} \right) \right]^2} \]

\[ = \frac{1}{1 + \frac{16n^4}{(n+1)^2}} < \frac{(n+1)^2}{16n^4} \leq \frac{1}{2n(n+1)}. \]
and for \( n \geq 2 \),
\[
S_n (b_{n-1}) = \frac{1}{1 + 64n^6 \left[ \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+1} \right) \right]^2} < \frac{1}{1 + \frac{16n^2}{(n-1)^2}} < \frac{1}{16n (n + 1)},
\]
while for \( n = 1 \),
\[
S_1 (b_0) = \frac{1}{1 + 64 (2 - 1)^2} \leq \frac{1}{(32) (1) (2)}.
\]
Thus
\[
\sum_{n=1}^{\infty} \max \{ S_n (b_n), S_n (b_{n-1}) \} < \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n (n + 1)} = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + 1} \right) = \frac{1}{2},
\]
and
\[
g(x) = \sum_{n=1}^{\infty} \frac{1}{1 + 64n^6 (x - \frac{1}{n})^2} = \sum_{n=1}^{\infty} \frac{1}{1 + 64n^4 (nx - 1)^2}
\]
does not have a limit as \( x \to 0^+ \).

Remark 1. A faster example is the Fourier series of \( f(x) \) restricted to \( [-\pi, \pi) \). However, the coefficients are not easily expressed in closed form and a little theory is needed to demonstrate convergence.

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