The Limit Comparison Test Needs Positivity

J. MARSHALL ASH
DePaul University
Chicago, IL 60614-3504
mash@math.depaul.edu

The limit comparison test for infinite series appears in almost every modern calculus textbook. One statement is this.

THEOREM 1. Assume

(1) \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \) and
(2) \( a_n > 0 \) and \( b_n > 0 \) for all \( n \).

Then \( \sum a_n \) and \( \sum b_n \) both converge or both diverge.

This would be a much more beautiful theorem if we could just drop hypothesis (2). Unfortunately, this is not possible, as the following example illustrates. Let \( a_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n^2} \) and \( b_n = \frac{(-1)^n}{\sqrt{n}} \); then \( \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} 1 + \frac{(-1)^n}{\sqrt{n}} = 1 \), so hypothesis (1) is true. However, \( \sum b_n \) converges and \( \sum a_n \) diverges.

In a certain sense, this is the only possible example.

THEOREM 2. If

(3) \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \) and
(4) \( \sum b_n \) converges and \( \sum a_n \) diverges,

then \( \sum b_n \) converges conditionally; and if we write \( a_n = b_n + c_n \) for all \( n \), then \( \sum c_n \) diverges and the \( \{c_n\} \) are "infinitely smaller" than the \( \{b_n\} \).

Proof: Assume \( \sum b_n \) converges absolutely. From (3), \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \). By the usual limit comparison test for positive series, \( \sum a_n \) is absolutely convergent. Consequently, \( \sum c_n \) is convergent, a contradiction. So \( \sum b_n \) is conditionally convergent. For each \( n \), define \( c_n := a_n - b_n \). If \( \sum a_n \) converges, then \( \sum a_n \) must also, so \( \sum c_n \) diverges. Finally, the \( \{c_n\} \) are infinitely smaller than the \( \{b_n\} \) in the sense that

\[
\lim_{n \to \infty} \frac{c_n}{b_n} = \lim_{n \to \infty} \frac{a_n - b_n}{b_n} = \lim_{n \to \infty} \frac{a_n}{b_n} - 1 = 0. \tag*{\blacksquare}
\]

So the most general possible counterexample involves two series, the "big" but convergent \( \sum b_n \), and the "little" but divergent \( \sum c_n \). Two natural questions are whether for every convergent \( \sum b_n \) we can find a corresponding \( \sum c_n \) to create a counterexample as above, and whether for every divergent \( \sum c_n \) we can find a corresponding \( \sum b_n \) to create a counterexample as above. To be more specific, we ask the following two questions.

(i) Given any convergent series \( \sum b_n \), does there exists a "poisoning" series \( \sum c_n \) that is small, \( c_n = o(b_n) \), and such that \( \sum (b_n + c_n) \) is divergent?
(ii) Given any divergent series \( \sum c_n \), does there exist a "healing" series \( \sum b_n \) that is big, \( c_n = o(b_n) \), and such that \( \sum b_n \) is convergent?

As we pointed out above, because of the limit comparison test, in question (i) \( \sum b_n \) must be conditionally convergent. In question (ii), the terms of \( \sum c_n \) must tend to 0, since otherwise \( c_n = o(b_n) \) would be impossible. We will show that these fairly obvious necessary conditions are also sufficient.

For question (i), let \( \sum b_n \) be any conditionally convergent series. Define \( p_n \) by

\[
p_n = \begin{cases} 
  b_n & \text{if } b_n > 0, \\
  0 & \text{otherwise}.
\end{cases}
\]

It is well known that \( \sum p_n = \infty [2, \text{p. 375}] \). Next let \( c_n \) be nonnegative, equal to zero when \( p_n \) is, and satisfy both \( c_n = o(p_n) \) and \( \sum c_n = \infty \). This can be done since there is no slowest positive divergent series \([1]\). Then \( \frac{c_n}{b_n} = 0 \) when \( p_n = 0 \), and \( \frac{c_n}{b_n} = o(\frac{c_n}{p_n}) \) when \( p_n > 0 \), so \( c_n = o(b_n) \).

For question (ii), let \( \sum c_n \) be any divergent series with terms tending to 0. Let \( c_n^* = \sup_{\nu \geq n} |c_\nu| \). We will construct \( \sum b_n \). Define \( p_0 \) to be the first index so that \( c_n^* p_0 \leq 4^{-1} \), \( p_1 \) to be the first index so that \( c_n^* p_1 \leq 4^{-2} \), \( p_2 \) to be the first index so that \( c_n^* p_2 \leq 4^{-3} \), \ldots. By increasing the \( p_i \) as necessary, we may assume that \( p_0 - 1 \) is a multiple of 2, \( p_1 - p_0 \) is a multiple of \( 2^2 \), \( p_2 - p_1 \) is a multiple of \( 2^3 \), \ldots. Break the set of indices \( n \) such that \( 1 \leq n < p_0 \) into blocks of length 2 and set the values of \( b_n \) to be 1, -1 on each block. Next, break the set of indices \( n \) such that \( p_0 \leq n < p_1 \) into blocks of length \( 2^2 \) and set the values of \( b_n \) to be \( -1, -2, -2, 2, 2, -2, -2, 2, 2, -2, -2, 2, 2, -2, -2 \) on each block. Proceed inductively. For each interval \([p_{i-1}, p_i)\), we have \( k_i \) blocks, each of length \( 2^{i+1} \), and

\[
\{|b_n|\}_{n=p_{i-1}}^{p_i-1} = \left\{ \frac{1}{2^i}, -\frac{1}{2^i}, \frac{1}{2^i}, -\frac{1}{2^i}, \ldots, \frac{1}{2^i}, -\frac{1}{2^i} \right\}.
\]

The sum of the \( b_n \) over each of the \( k_i \) blocks is 0, while the corresponding sum of the \( |b_n| \) is 2. Then \( \sum b_n \) converges to 0 and \( \sum |b_n| = 2k_1 + 2k_2 + \cdots + 2k_i + \cdots \), so that \( \sum b_n \) converges conditionally. For each \( i \geq 1 \) and each index \( n \) such that \( p_{i-1} < n < p_i \), we have \( |c_n| \leq 4^{-i} \) and \( |b_n| = 2^{-i} \) so that

\[
\frac{|c_n|}{|b_n|} \leq \frac{4^{-i}}{2^{-i}} = \frac{1}{2}
\]

and \( c_n = o(b_n) \) as required.

The author was able to find an instance of this family of examples in the literature. However, this has definitely been in the mathematical folklore for a long time. For example, on page 376 of [2], G. H. Hardy, remarks explicitly “…there are no comparison tests for convergence of conditionally convergent series.” It seems likely that he had one of these examples in mind to make such a categorical statement.

REFERENCES


Summary The limit comparison test for positive series does not extend to general series. An example is given. In a certain sense, this is the only possible example. Given a conditionally convergent series, there exists a termwise much smaller series so that the sum of the two series diverges. Given a divergent series with terms tending to zero, there exists a convergent but termwise much bigger series.