Victor Shapiro and the theory of uniqueness for multiple trigonometric series

J. Marshall Ash

Abstract. In 1870, Georg Cantor proved that if a trigonometric series converges to 0 everywhere, then all its coefficients must be 0. In the twentieth century this result was extended to higher dimensional trigonometric series when the mode of convergence is taken to be spherical convergence and also when it is taken to be unrestricted rectangular convergence. We will describe the path to each result. An important part of the first path was Victor Shapiro's seminal 1957 paper, *Uniqueness of multiple trigonometric series.* This paper also was an unexpected part of the second path.

1. Dedication

My thesis advisor, Antoni Zygmund, and his student, Albert Calderón, created a wonderful school of mathematical analysis that radiated outward from the University of Chicago in the 1950s and 1960s. Their intellectual curiosity, complemented by their generosity and friendliness, induced similar good feelings among their mathematical descendents. This is a large community. For example, there were 66 direct descendents, since Zygmund had 40 Ph.D. students, Calderón had 27, and there is one student, Cora Sadosky, who had both listed for thesis advisors.[**MG**]

News of the loss of the cheerful, enthusiastic mathematician Cora Sadosky in 2010 was sad news indeed. Victor Shapiro was a student of Zygmund, and therefore a fellow "mathematical sibling" of Cora and mine. His death in 2013 was followed later in that year by a conference to honor him held in Riverside, California. I gave a talk at that conference. The paper that I present here is based on that talk. I suspect that Cora, whose first question upon meeting me after the passage of some time was usually about my mathematics, would have enjoyed seeing this.

1991 *Mathematics Subject Classification.* Primary 42-02; Secondary 42A63, 42B99, 42A20 .

Key words and phrases. Cantor's uniqueness theorem, Uniqueness.

This research was partially supported by a grant from the Faculty and Development Program of the College of Liberal Arts and Sciences, DePaul University.

2. Two theorems and a conjecture

Let $\{d_n\}_{-\infty<n<\infty}$ be a sequence of complex numbers and let $x \in \mathbb{T}^1 = [0, 2\pi)$. Suppose a function has a representation of the form

$$\sum d_n e^{inx} = \lim_{N\to\infty} d_0 + \sum_{n=1}^{N} \left(d_{-n}e^{-inx} + d_n e^{inx}\right).$$

It is natural to combine the nth and $-n$th terms, for if a_n and b_n are real, $d_n = (a_n + ib_n)/2$ and d_{-n} is the complex conjugate of d_n, then. $d_n e^{inx} + d_{-n}e^{-inx} = a_n \cos nx + b_n \sin nx$, the "natural" nth term of a real valued trigonometric series. Is this representation unique? In other words, if $\sum d_n e^{inx} = \sum d'_n e^{inx}$ for every x, does it necessarily follow that $d_n = d'_n$ for every n? Subtract and set $c_n = d_n - d'_n$ to get a cleaner formulation: Does $\sum c_n e^{inx} = 0$ imply that $c_n = 0$ for every n? Here is Georg Cantor's answer.

Theorem 1. *Let $\sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^1$. Then $c_n = 0$ for every n.*

He proved this in 1870. Notice that in the statement of his theorem, Cantor made a choice of what it means for a trigonometric series to represent the function $z(x)$ where z has domain $\mathbb{T}^1$ and range $\{0\}$, namely that it converge to that point at every point of $\mathbb{T}^1$. Many other notions of "represent" have been considered since then; many are discussed in chapter IX of Antoni Zygmund's book Trigonometric Series.[**Z1**] We will mostly focus on this pointwise everywhere notion of representation.

The entire subject of this broad survey concerns attempts to extend this result to higher dimensions. In all dimensions we will always combine terms whose indices differ only by signs. This reduction in dimension 1 converts a two-sided numerical series $\sum_{n=-\infty}^{\infty} C_n$ to the series $\sum_{n\in\mathbb{Z}^+} T_n$, where for each $n \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, $T_n = \sum_{\{\nu:\ |v|=n\}} C_\nu$. Since the nonnegative integers have a natural ordering, Cantor's theorem's hypothesis is unambiguous. When $d \geq 2$, the corresponding reduction of $\sum_{n\in\mathbb{Z}^d} C_n$ to $\sum_{n\in(\mathbb{Z}^+)^d} T_n$ where $T_n = \sum_{\{\nu:\ |\nu_i|=n_i \text{ for } 1\leq i\leq d\}} C_\nu$ does not produce a "natural ordering" because $(\mathbb{Z}^+)^d$ does not have a natural ordering, so many conjectures arise in each dimension.

Here are three important distinct ways of adding the elements of the numerical series $\sum_{n\in(\mathbb{Z}^+)^d} T_n$.

Spherical convergence: The Nth partial sum contains all terms with indices in the intersection of the sphere of radius $\sqrt{N}$ with the positive cone $(\mathbb{Z}^+)^d$. The spherical sum is defined to be

$$SPH \sum_{n\in(\mathbb{Z}^+)^d} T_n = \lim_{N\to\infty} \sum_{\left\{\nu:\text{all } \nu_i\geq 0 \text{ and } \nu_1^2+\cdots+\nu_d^2\leq N\right\}} T_\nu.$$

Square convergence: The Nth partial sum contains all terms with indices in the rectangular parallelepiped with opposite corners $(0, \dots, 0)$ and $(N, \dots, N)$. The square sum is defined as

$$SQ \sum_{n\in(\mathbb{Z}^+)^d} T_n = \lim_{N\to\infty} \sum_{\nu_1=0}^{N} \cdots \sum_{\nu_d=0}^{N} T_\nu.$$

Unrestricted rectangular convergence: This is no longer a one variable process. Assign to each point n of $(\mathbb{Z}^+)^d$ the rectangular partial sum S_n, of all terms whose indices are in the rectangular parallelepiped with corners $(0,\ldots,0)$ and $(n_1,\ldots,n_d)$. The unrestricted rectangular limit of $\sum T_\nu$ is a number L such that for each $\epsilon > 0$, there is a number $N(\epsilon)$ so that for every n with $\min\{n_1,\ldots,n_d\} > N(\epsilon)$, $|S_n - L| < \epsilon$. When such an L exists, we call it the unrestricted rectangular sum of $\sum T_n$ and write

$$UR \sum_{n\in(\mathbb{Z}^+)^d} T_n = \lim_{\min\{N_1,\cdots,N_d\}\to\infty} \sum_{\nu_1=0}^{N_1}\cdots\sum_{\nu_d=0}^{N_d} T_\nu .$$

It is obvious that if a numerical series is unrestrictedly rectangularly convergent, then it is square convergent to the same sum; i.e.,

$$UR \sum_{n\in(\mathbb{Z}^+)^d} T_n = L \text{ implies } SQ \sum_{n\in(\mathbb{Z}^+)^d} T_n = L. \tag{2.1}$$

It is easy to give examples of double series of numbers $\sum_{n\in(\mathbb{Z}^+)^2} T_n$ which show that each of the five other possible connections between these three methods of convergence is, in general, false.

In dimension $d \geq 2$, with $n = (n_1,\ldots,n_d) \in \mathbb{Z}^d$, $x = (x_1,\ldots,x_d) \in \mathbb{T}^d$, and $nx = n_1x_1 + \cdots + n_dx_d$, and $T_n(x) = \sum_{\{\nu:\ |v|=n\}} c_\nu e^{i\nu x}$, there are three distinct and natural hypotheses for direct generalizations of Cantor's Theorem. Here is the present state of knowledge.

Theorem 2. *Let* $SPH \sum_{n\in\mathbb{Z}^d} c_n e^{inx} = SPH \sum_{n\in(\mathbb{Z}^d)^+} \sum_{\{\nu:\ |v|=n\}} c_\nu e^{i\nu x} = 0$ *for every* $x \in \mathbb{T}^d$. *Then* $c_n = 0$ *for every* n.

Theorem 3. *Let* $UR\sum_{n\in\mathbb{Z}^d} c_n e^{inx} = UR \sum_{n\in(\mathbb{Z}^d)^+} \sum_{\{\nu:\ |v|=n\}} c_\nu e^{i\nu x} = 0$ *for every* $x \in \mathbb{T}^d$. *Then* $c_n = 0$ *for every* n.

Conjecture 1. *Let* $SQ\sum_{n\in\mathbb{Z}^d} c_n e^{inx} = SQ \sum_{n\in(\mathbb{Z}^d)^+} \sum_{\{\nu:\ |v|=n\}} c_\nu e^{i\nu x} = 0$ *for every* $x \in \mathbb{T}^d$. *Then* $c_n = 0$ *for every* n.

We will later need to mention one higher dimensional extension of Theorem 2 which involves replacing the condition of spherical convergence: $SPH \sum c_n e^{inx}$ exists, by the weaker condition of spherical Abel summability:

Theorem 4. *If for every* $x \in \mathbb{T}^d$, $SPH \sum c_n e^{inx} r^{\|n\|}$ *exists for all positive* $r < 1$, *where* $\|n\| = \sqrt{n_1^2 + \cdots + n_d^2}$, *and* $\lim_{r\to 1^-} SPH \sum c_n e^{inx} r^{\|n\|} = 0$, *and if*

$$\sum_{R-1<\|n\|\leq R} |c_n| = o(R) \text{ as } R \to \infty; \tag{2.2}$$

then $c_n = 0$ *for every* n.

3. History of the two theorems

The steps of Cantor's brilliant proof are well known. Our discussion here will be informed by drawing comparisons with them. Here are the four major steps of his proof.

(1) Establish the Cantor-Lebesgue Theorem, which implies that everywhere convergence ensures that

$$\epsilon(R) = \sum_{\|n\|=R} |c_n|^2 \to 0 \text{ as } R \to \infty. \tag{3.1}$$

In dimension 1, $\sum_{\|n\|=R} |c_n|^2 = |c_R|^2 + |c_{-R}|^2$.

(2) Show that the Riemann function, the formal second integral, $F(x) = c_0 \frac{x^2}{2} + \sum_{n\neq 0} \frac{c_n}{(in)^2} e^{inx}$, is continuous. (Formal means that $\frac{d^2}{dx^2} \frac{x^2}{2} = 1$ and for each $n \neq 0$, $\frac{d^2}{dx^2} \frac{e^{inx}}{(in)^2} = e^{inx}$.)

(3) Establish the consistency of Riemann summability, that the Schwarz second derivative D^2 defined by

$$D^2 F(x) = \lim_{h\to 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} \tag{3.2}$$

satisfies at every x

$$D^2 F(x) = \lim_{h\to 0} c_0 + \sum_{n\neq 0} c_n e^{inx} \left(\frac{\sin n\frac{h}{2}}{n\frac{h}{2}} \right)^2 = 0.$$

(4) Use Schwarz's Theorem, that continuous functions with identically zero Schwarz second derivative are of the form $ax + b$.

Step (2), the proof that F is continuous, is immediate from Step (1) and the Weierstrass M-Test.

3.1. The spherical uniqueness theorem, Theorem 2. The first big step towards Theorem 2 was taken in 1957, when Victor Shapiro proved a powerful d dimensional theorem. Shapiro worked in a more general context, also considering questions of summability. He did not prove Theorem 2 because his proof required an extra assumption on the coefficient size.[**S**] A corollary of one of Shapiro's results was a weaker version of Theorem 2 which required the additional hypothesis of condition (2.2). This condition is quite natural when $d = 2$: since there are $O(r)$ lattice points being summed over in condition (2.2), in dimension 2 this assumption asserts that the c_m tend to zero "on the average" as $|m| \to \infty$. But the assumption becomes much stronger as the dimension increases; specifically in dimension d there are $O\left(r^{d-1}\right)$ terms in the sum, so that the coefficients are required to be decaying like $o\left(r^{2-d}\right)$ on the average.

Shapiro's 1957 proof is a direct extension of Cantor's in the sense that he follows the same four steps.

(1) He controls the coefficient size by simply adding a second hypothesis, namely that condition (2.2) holds is true *by assumption*.

(2) His Riemann function is the formal anti-Laplacian

$$F(x) = c_0 \frac{\|x\|^2}{2d} - \sum_{n\neq 0} \frac{c_n}{\|n\|^2} e^{inx} = \lim_{r\to 1^-} c_0 \frac{\|x\|^2}{2d} - \sum_{n\neq 0} \frac{c_n}{\|n\|^2} e^{inx - \|n\| t},$$

which he proves to be continuous. (Now formal means that $\Delta \frac{\|2x\|^2}{2d} = 1$ and for each $n \neq 0$, $\Delta \frac{e^{inx}}{\|n\|^2} = e^{inx}$.)

(3) He establishes a kind of consistency of Riemann summability by showing that at every x, the generalized Laplacian of F,

$$\lim_{h\to 0^+}\frac{8}{h^2}\left\{\frac{1}{\pi h^2}\int_{\|\eta\|<h}F\left(x+\eta\right)d\eta-F\left(x\right)\right\}$$

is zero.

(4) Use a well known theorem that continuous functions with identically zero generalized Laplacian are harmonic.

By far the most delicate and difficult part of his work is Step (2), the proof that F is continuous.

Because the original series is only required to be Abel summable to 0 everywhere, it is not possible to weaken the hypothesis by replacing $o\left(R\right)$ by $0\left(R\right)$. For the proposed stronger theorem would be contradicted by the fact that the one dimensional series

$$\delta'\left(x\right)=\sum ine^{inx}=-2\sum n\sin nx$$

has Abel limit 0 everywhere.

To my mind, the most beautiful thing about Theorem 2 is that there is no hypothesis about coefficient size. In 1971, 14 years after Shapiro's theorem was proved, Roger Cooke proved a two dimensional Cantor-Lebesgue theorem.[**Coo**] An immediate consequence of his result is that everywhere two dimensional spherical convergence ensures that condition (3.1) must hold. But in dimension 2 (and *not* in higher dimensions), condition (3.1) implies condition (2.2). So the two dimensional version of Theorem 2 was proved.

The first precursor to a higher dimensional spherical theory came in 1976, when Bernard Connes extended the Cantor Lebesgue result of Cooke, whose proof was exceedingly two dimensional, to all dimensions.[**Con**] At this point, we knew that if we wanted to prove Theorem 2, we could use the fact that $\epsilon\left(R\right)\to 0$ without having to add a second hypothesis involving coefficient size.

But in dimension 3, there is a very large gap between condition (3.1) and the much stronger condition (2.2), and this gap becomes ever larger as the dimension increases. So it seemed likely that when dimension ≥ 3, Step (2) of Shapiro's proof, the proof that his Riemann function is continuous, would be inaccessible. Shapiro and I discussed this problem. We agreed that it was totally unclear if there was a proof or a counterexample ahead for the cases of $d\geq 3$, and he speculated that there might be a century of mathematical analysis development required to bring this question within reach.

Victor Shapiro had one more major contribution to make toward the solution of Theorem (2). Victor told me that one day in the middle 1990s, he and Jean Bourgain happened to be strolling across the University of California, Riverside campus and Victor mentioned this problem. Their conversation inspired Bourgain to look at the problem and solve it!. He stayed entirely within the framework established by Cantor and generalized by Shapiro. Assuming only the everywhere spherical convergence to zero, and bringing together Connes result, hard analysis, harmonic measure, and some probability theory (martingales), he was able to prove that Shapiro's Riemann function was continuous. In 1996, Bourgain published his proof that the hypothesis of Theorem 2 implies the continuity of Shapiro's Riemann function.[**B**] Together, these two excellent papers, the first by Shapiro and the latter by Bourgain, published 39 years apart, provide the complete proof of Theorem 2.

Bourgain's paper is only 15 pages long. Although it is absolutely correct and says everything that should be said in just the right order, it is extremely terse. In fact, Gang Wang and I took nine months to read it, but once we got it, we were able to reproduce a lot of the substantial collection of one dimensional extensions of Cantor's Theorem that can be found in chapter IX of Antoni Zygmund's *Trigonometric Series*.[**AWa1**, **AWa2,Z1**] To see an expansion of Bourgain's proof of continuity, see the 22 page version in [**AWa1**]; and to see his proof expanded to 42 pages while being specialized down to two dimensions, see [**As3**].

3.2. The unrestricted rectangular uniqueness theorem, Theorem 3. Let $M(x)$ be a one dimensional trigonometric series that converges to zero a.e. and let $\delta(y) = \sum e^{iny}$ be the trigonometric series associated with the unit mass at the origin. Because the partial sums of $\delta(y)$ are bounded by $\csc y$, the double trigonometric series $M(x)\delta(y)$ is unrestrictedly rectangularly convergent to 0 a.e.[**AWe**] But by Zygmund's extension of Cooke's theorem, $M\delta$ cannot converge circularly on a set of positive measure, since its coefficients do not tend to 0 as $\|n\| \to \infty$.[**Z2**] So the proof of Theorem 3 seems to have nothing to do with Shapiro's 1957 theorem. However, by a strange quirk of fate, it does.

Theorem 3 was announced in 1919. The announced proof also followed the model of Cantor's Theorem. It was a very simple induction that seemed to indicate that there were no interesting things to do in this direction, so for many years nothing involving uniqueness for unrestricted rectangular convergence appeared in the literature. Grant Welland and I studied the proof around 1970 and could not follow the step that generalized Schwarz's Theorem. In fact, some years later Chris Freiling and Dan Rinne showed me the counterexample function $(x+y)|x+y|$. This function satisfies the hypotheses(being continuous and having a certain generalized F_{xxyy} identically 0), but not the conclusion(having the expected form

$$a(y)x + b(y) + c(x)y + d(x),$$

with a, b, c, d being twice differentiable), of the generalized Schwarz's Theorem necessary for the 1919 paper's proof to be valid.

So Grant Welland and I tried to prove Theorem 3. We were able to prove a version of the Cantor-Lebesgue theorem stating that when a multiple trigonometric series converges unrestrictedly rectangularly a.e., "most" coefficients tend to zero, while all coefficients are bounded. From this control of the coefficient size, it follows that Shapiro's coefficient size condition 2.2 holds in dimension 2. In view of the $M\delta$ example just mentioned, one cannot expect the hypothesis of

$$UR \sum_{n \in \mathbb{Z}^d} c_n e^{inx} = 0 \text{ for all } x \in \mathbb{T}^d \tag{3.3}$$

to easily imply that $SPH \sum_{n\in\mathbb{Z}^d} c_n e^{inx} = 0$ for all $x \in \mathbb{T}^d$. Nevertheless, it turns out to be easy to prove that hypothesis (3.3) does imply spherical Abel summability to zero everywhere. (This was quite an unexpected and happy surprise for Welland and me.) Thus Shapiro's more general Theorem 4 does apply here, so that in 1972 the two dimensional case of Theorem 3 was shown to be another consequence of Shapiro's 1957 results.[**AWe**]

After a gap of about 20 years, during which there was no activity at all in the area of uniqueness for multiple trigonometric series, two completely different proofs of Theorem 3 for all dimensions appeared.[**AFR**, **T**] The Tetunashvili proof involves a clever induction. Some ideas from [**Coh**] and [**AWe**] play a role. The Ash-Freiling-Rinne proof extensively renovates the 1919 attempted proof, and uses a complicated covering argument. (See [**As1**] to see this covering argument applied in a much simpler situation.) It is ironical that our very complicated covering proof probably would not have happened if Tetunashvili's previously published and more direct proof had come to our attention before our article had appeared. Only time will tell if the covering techniques we developed will eventually have useful applications elsewhere.

4. The conjecture

There is an obvious inclusion. If a multiple numerical series converges Unrestricted Rectangularly, then it converges Square. So

$$\text{Hypothesis of UR Theorem} \implies \text{Hypothesis of Square Conjecture.}$$

This explains how it can be that UR uniqueness can be known while Square uniqueness remains an open question. One hint of the problems here is that it is possible for a double trigonometric series to square converge everywhere to a finite valued function while having coefficients that are not $O\left(n^J\right)$ no matter how large J may be.[**AWa2**] I have discussed the square uniqueness conjecture in several places.[**As2**]

The first seven references below can be found using links from `http://condor.depaul.edu/mash/realvita.html.`

References

[As1] J. M. Ash, *A new, harder proof that continuous functions with Schwarz derivative 0 are lines,* Contemporary Aspects of Fourier Analysis, W. O. Bray and C. V. Stanojevic,(Eds.), 1994, pp. 35-46.

[As2] J. M. Ash, *Uniqueness for higher dimensional trigonometric series,* Cubo, **4**(2002), 97-125.

[As3] J. M. Ash, *Uniqueness for Spherically Convergent Multiple Trigonometric Series,* Handbook of Analytic-Computational Methods in Applied Mathematics, G. Anastassiou ed., Chapman & Hall/CRC, Boca Raton, 2000, pp. 309-355.

[AFR] J. M. Ash, C. Freiling, and D. Rinne, *Uniqueness of rectangularly convergent trigonometric series,* Ann. of Math. **137** (1993), 145–166.

[AWa1] J. M. Ash and G. Wang, *A survey of uniqueness questions in multiple trigonometric series*, A Conference in Harmonic Analysis and Nonlinear Differential Equations in Honor of Victor L. Shapiro, Contemporary Mathematics, **208** (1997), 35-71.

[AWa2] J. M. Ash and G. Wang, *One and two dimensional Cantor-Lebesgue type theorems,* Trans. Amer. Math. Soc., **349**(1997), 1663-1674.

[AWe] J. M. Ash and G. V. Welland, *Convergence, uniqueness, and summability of multiple trigonometric series*, Trans. Amer. Math. Soc. **163** (1972), 401–436. MR **45** # 9057.

[B] J. Bourgain, *Spherical summation and uniqueness of multiple trigonometric series,* Internat. Math. Res. Notices (1996), 93-107.

[Coh] P. J. Cohen, "*Topics in the theory of uniqueness of trigonometrical series,*" doctoral thesis, University of Chicago, Chicago, IL., 1958.

[Con] B. Connes, *Sur les coefficients des séries trigonometriques convergents sphériquement*, CRA Sc Paris t. Ser. A **283** (1976), 159-161.

[Coo] R. L. Cooke, *A Cantor-Lebesgue theorem in two dimensions,* Proc. Amer. Math. Soc., **30**(1971), 547–550. MR **43** #7847.

[MG] Mathematics Genealogy Project, http://genealogy.math.ndsu.nodak.edu/

[S] V. L. Shapiro, *Uniqueness of multiple trigonometric series,* Ann. of Math. **66**(1957), 467–480. MR **19**, 854; 1432.

[T] S. Tetunashvili, *On some multiple function series and the solution of the uniqueness problem for Pringsheim convergence of multiple trigonometric series,* Mat. Sb. **182** (1991), 1158–1176 (Russian), Math. USSR Sbornik **73** (1992), 517–534 (English). MR 93b:42022.

[Z1] A. Zygmund, Trigonometric Series, v.1, 2nd ed., 1958.

[Z2] A. Zygmund, *A Cantor-Lebesgue theorem for double trigonometric series,* Studia Math, **43** (1972), 173–178. MR **47** #711.

Mathematics Department, DePaul University, Chicago, IL 60614
E-mail address: mash@math.depaul.edu
URL: http://www.depaul.edu/~mash