Example 1.2: (See note [1]): Let p be a positive integer. Solving the following problem, we immediately know that we should put $q = \frac{1}{p+1}$ in Example 1.1.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt[p+1]{k^p+(k+1)^p+\ldots+n^p}}{n^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{\left(\frac{k}{n}\right)^p+\left(\frac{k+1}{n}\right)^p+\ldots+\left(\frac{n}{n}\right)^p}{n}\right\}^{\frac{1}{p+1}}$$

$$= \frac{\left\{\Gamma\left(\frac{1}{p+1}+1\right)\right\}^2}{\sqrt[p+1]{p+1}\,\Gamma\left(\frac{2}{p+1}+1\right)}.$$

In a similar way, we can prove the following theorem.

Theorem 2:

Let $f(x) \geqslant 0$ be a continuous and increasing function in $[0, \infty)$. If $F'(x) = f(x)$ and q is a constant with $0 < q \leqslant 1$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{k}{n}\right)}{n}\right\}^q = \int_0^1\{F(x)-F(0)\}^q\,dx.$$

Example 2: (See note [1]): Let p be a positive integer. Then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt[p+1]{1^p+2^p+\ldots+k^p}}{n^2} = \frac{1}{n}\lim_{n\to\infty}\sum_{k=1}^{n}\left\{\frac{\left(\frac{1}{n}\right)^p+\left(\frac{2}{n}\right)^p+\ldots+\left(\frac{k}{n}\right)^p}{n}\right\}^{\frac{1}{p+1}}$$

$$= \int_0^1\left(\frac{x^{p+1}}{p+1}\right)^{\frac{1}{p+1}}dx = (p+1)^{-\frac{1}{p+1}}\int_0^1 x\,dx = \frac{1}{2\times\sqrt[p+1]{p+1}}.$$

Reference

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93.55 Constructing a quadrilateral inside another one

1. *The quadrilateral ratio problem*

In a convex quadrilateral $ABCD$ join A to the midpoint of $\overline{BC}$, B to the midpoint of $\overline{CD}$, C to the midpoint of $\overline{DA}$, and D to the midpoint of $\overline{AB}$. The intersections of these segments determine an inner quadrilateral $EFGH$ as shown in Figure 1.

Let r be the ratio of the quadrilateral areas, i.e.

$$r = \frac{\text{area}\,(EFGH)}{\text{area}\,(ABCD)}. \tag{1}$$

The problem is to describe the value of r in terms of the geometry of $ABCD$.

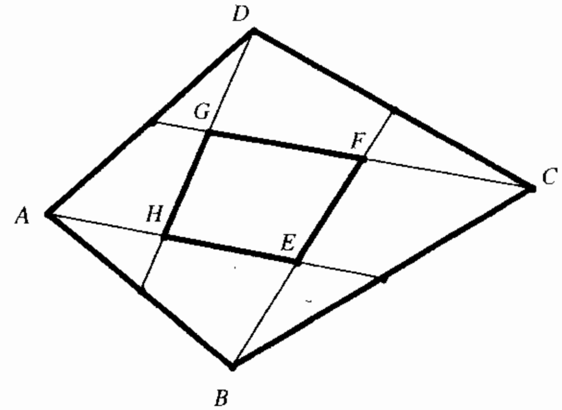

FIGURE 1

We first saw this problem in a projects book for Geometer's Sketchpad [1]. We later learned that this problem is due to Michael de Villiers [2]. Simple numerical exploration has led some students to conjecture that $r = \frac{1}{5}$. In Theorem 1 we show that this is true in the case when the original quadrilateral is a parallelogram. However, the conjecture is false in general. In fact, the ratio can be any real number in the interval $\left(\frac{1}{6}, \frac{1}{5}\right]$. This is part of our Corollary 3, which provides a complete solution to the problem.

Suppose now that instead of a quadrilateral we had a triangle. Of course, joining each vertex to the opposite midpoint would not yield an inner triangle, since the three lines are medians, which are concurrent in a point. To look for an analogous result for a triangle, we can look for points which are not midpoints, but rather divide each side a ratio ρ of the distance from one point to the next, $0 < \rho < 1$. For definiteness, we assume that 'next point' in this definition is based on movement in a counterclockwise direction. We call these points *ρ-points*. It turns out, that for a given ρ, the ratio of the area of the inner triangle to the area of the outer triangle is a constant independent of the initial triangle and is given by $\frac{(2\rho-1)^2}{\rho^2-\rho+1}$. Note that when $\rho = \frac{1}{2}$, this reduces to 0, which provides a convoluted proof that the medians of a triangle are concurrent. When $\rho = \frac{1}{3}$, the area ratio is $\frac{1}{7}$, which the Nobel Prize-winning physicist Richard Feynman once proved, though he was probably not the first to do so [3]. The result for general ρ is known, and a proof is given in [4], along with the Feynman story.

Inspired by this result, we shall study the quadrilateral question for ρ-points. Let $ABCD$ be a convex quadrilateral and let N_1, N_2, N_3 and N_4 be chosen so that N_1 is the ρ-point of BC, N_2 is the ρ-point of CD, N_3 is the ρ-point of DA, and N_4 is the ρ-point of AB. For fixed ρ $(0 < \rho < 1)$ connect each vertex of $ABCD$ to the ρ-point of the next side (A to N_1, B to N_2, C to N_3, and D to N_4). The intersections of the four line segments form the vertices of a quadrilateral $EFGH$.

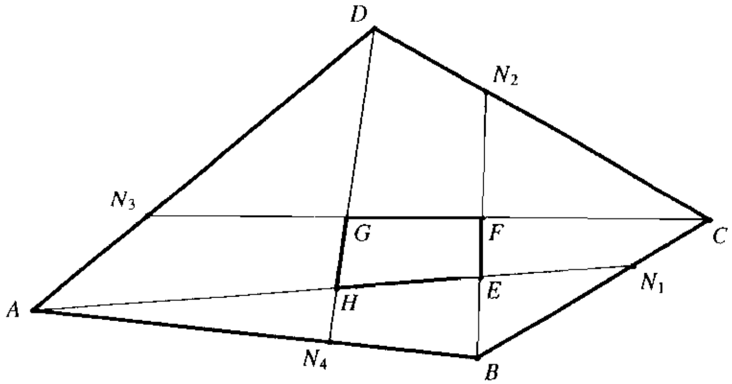

FIGURE 2

Define the area ratio

$$r(\rho, ABCD) = \frac{\text{area}(EFGH)}{\text{area}(ABCD)}. \tag{2}$$

Theorem 2 below states that, as $ABCD$ varies, the values of $r(\rho, ABCD)$ fill the interval

$$(m, M] = \left(\frac{(1-\rho)^3}{\rho^2 - \rho + 1}, \frac{(1-\rho)^2}{\rho^2 + 1}\right]$$

and that it is possible to give an explicit characterisation of the set of convex quadrilaterals with maximal ratio M. The fact that $M - m$ has a maximum value of about 0.034 and is usually much smaller explains the near constancy of $r(\rho, ABCD)$ as $ABCD$ varies. Here are the graphs of M and m.

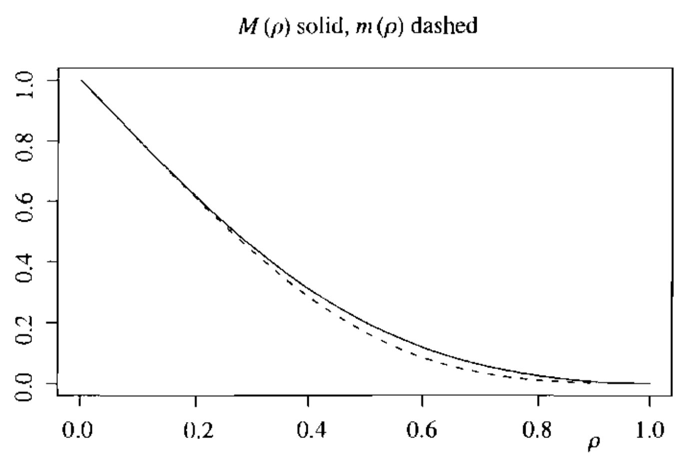

FIGURE 3

Here now is the graph of $M - m$.

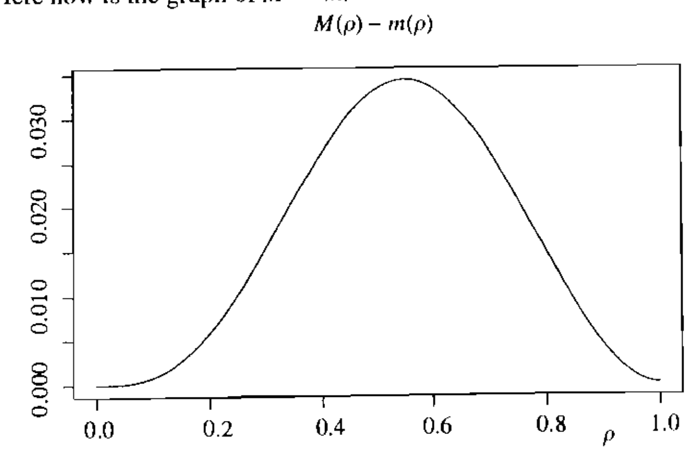

FIGURE 4

A more delicate look at the graph of $M - m = \dfrac{\rho^3(1-\rho)^2}{(\rho^2+1)(\rho^2-\rho+1)}$ shows that as 'constant' as r is in the original $\rho = \frac{1}{2}$ case, it is even 'more constant' when ρ is close to the endpoints 0 and 1. (Actually the maximum value of $M - m$ of about 0.034 is achieved at the unique real zero of $\rho^5 - \rho^4 + 6\rho^3 - 6\rho^2 + 7\rho - 3$ which is about 0.55.)

The characterisation proved in Theorem 2 below shows that not only do parallelograms have maximal ratio $M(\rho)$ for *every* ρ, but also they are the only quadrilaterals that have maximal ratio $M(\rho)$ for more than one ρ.

2. *The midpoint case for parallelograms*

Theorem 1: If each vertex of a parallelogram is joined to the midpoint of an opposite side in clockwise order to form an inner quadrilateral, then the area of the inner quadrilateral is one-fifth of the area of the original parallelogram.

Proof

In Figure 5, $ABCD$ is a parallelogram, and each M_i is a midpoint of the line segment it lies on. Cut apart the figure along all lines. Rotate ΔM_3GD clockwise 180° about the point M_3 to get the quadrilateral $AHGG'$, where G' is the image of G under the rotation. Quadrilateral $AHGG'$ is a parallelogram because the line CM_3 is parallel to the base of ΔAHD so that $DG = GH$. Parallelogram $AHGG'$ is congruent to $EFGH$, since similar reasoning shows that $AH = HE$. Triangle AHD has been rearranged into a parallelogram congruent to $EFGH$. Similarly, each of the triangles ABE, BCF and CDG may be dissected and rearranged to form a parallelogram congruent to $EFGH$. Thus, the pieces of $ABCD$ can be rearranged into five congruent parallelograms, one of which is $EFGH$, which therefore has area one-fifth of the area of $ABCD$.

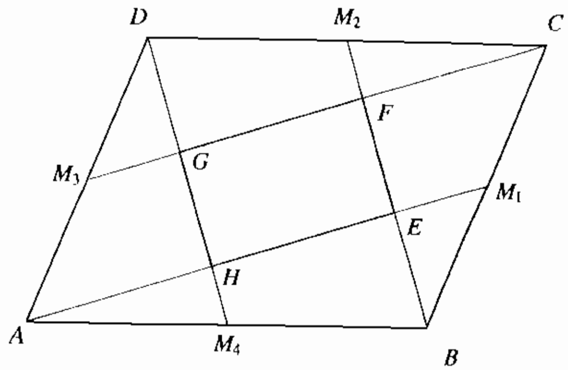

FIGURE 5

This result is a special case of Corollary 4 below, but is included because of the elegant and elementary nature of its proof.

3. *The filling of* $(m, M]$ *and the characterisation*

Theorem 2: Let A, B, C, D be (counterclockwise) successive vertices of a convex quadrilateral. Let r be the ratio defined by (2). Construct the point P so that $ABCP$ is a parallelogram. Locate (as in Figure 6) Q on $\overline{AB}$ so that $AQ = \rho AB$, and R on $\overrightarrow{BC}$ so that R is a distance ρBC from C and C is between B and R, and let $S = \overrightarrow{QP} \cup \overrightarrow{RP}$. Then the ratio r is maximal exactly when D is on $S^* = S \cap \operatorname{int}(\angle ABC) \cap \operatorname{ext}(\triangle ABC)$. The set of possible ratios is

$$\left(\frac{(1-\rho)^3}{\rho^2 - \rho + 1}, \frac{(1-\rho)^2}{\rho^2 + 1}\right].$$

Also the interior quadrilateral is a trapezium (with at least two parallel sides) if and only if the ratio is $(1-\rho)^2/(\rho^2+1)$.

In Figure 6 below, S^* is indicated by the thickened portions of the rays composing S.

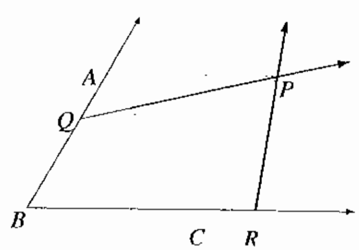

FIGURE 6

Proof: Fix ρ and apply an affine transformation that maps A, B, C, D to $(0, 1)$, $(0, 0)$, $(1, 0)$, (x, y) respectively. Since an affine transformation preserves both linear length ratios and area ratios, it is enough to prove the theorem after the transformation has been applied. Observe that P has become $(1, 1)$ and the image of S^* has become the union of two mutually perpendicular linear sets that intersect at $(1, 1)$, one a ray of slope ρ, and the other a line segment of slope $-1/\rho$. Here is the situation.

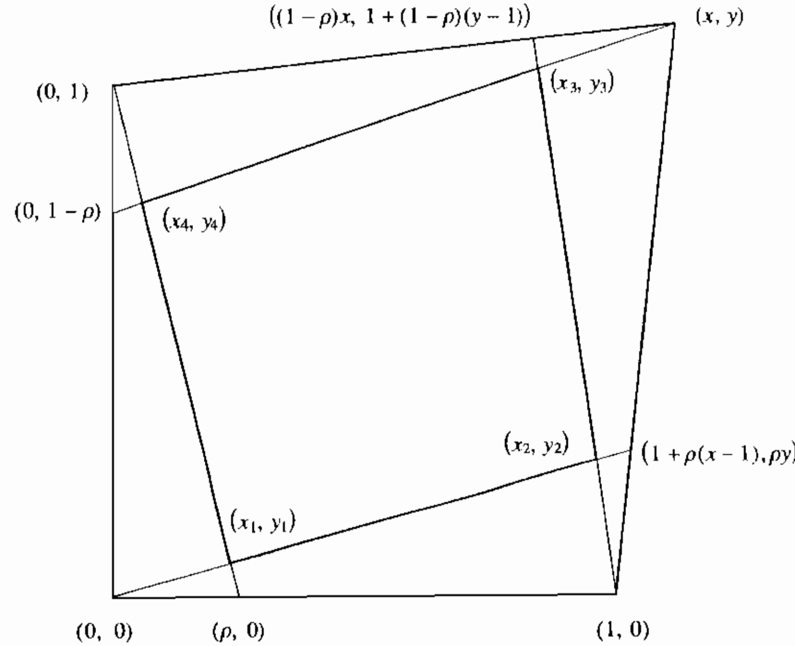

FIGURE 7

The line from $(0, 0)$ to (x, y) divides the outer quadrilateral into two triangles, one of area $\frac{1}{2}x$ and the other of area $\frac{1}{2}y$, so that its area is $\frac{1}{2}(x+y)$. To find the area of the inner quadrilateral, we first determine $\{x_1, \ldots, x_4, y_1, \ldots, y_4\}$ in terms of x, y and ρ by equating slopes. For example, the equations

$$\frac{y_1 - 0}{x_1 - 0} = \frac{\rho y - 0}{1 + \rho(x-1) - 0}$$
$$\frac{y_1 - 0}{x_1 - \rho} = \frac{1 - 0}{0 - \rho}$$

can easily be solved for x_1 and y_1. The area of the interior quadrilateral is

$$\frac{(x_1 - x_3)(y_2 - y_4) - (y_1 - y_3)(x_2 - x_4)}{2}.$$

This is the $n = 4$ case of a well-known formula for the area of an n-gon [5] which can be established by first proving the formula for triangles and then using induction, or by using Green's Theorem. Some computer algebra produces the following formidable and seemingly intractable formula for $r(x, y)$, the ratio of the interior and exterior quadrilateral areas:

$$
-(\rho-1)^2 \frac{\left|\begin{array}{c}
\rho^3(\rho-1)y^4 - \rho^2(3\rho^3 - 2\rho^2 - \rho + 2)xy^3 + \\
\rho(\rho^5 - \rho^4 - 6\rho^3 + 4\rho^2 - \rho - 1)x^2y^2 + \\
\rho(2\rho^4 - 2\rho^3 - 3\rho^2 + 2\rho - 1)x^3y + \rho^3(\rho-1)x^4 + \\
\rho(2\rho^4 - 6\rho^3 + \rho^2 + 2\rho - 1)y^3 - \\
(3\rho^6 - 10\rho^5 - 3\rho^4 + 13\rho^3 - 5\rho^2 - \rho + 1)xy^2 - \\
(3\rho^6 + 5\rho^5 - 15\rho^4 + \rho^3 + 7\rho^2 - 4\rho + 1)x^2y - \\
\rho^2(3\rho^3 + 2\rho^2 - 5\rho + 2)x^3 + \\
\rho(\rho^5 - 7\rho^4 + 6\rho^3 + 7\rho^2 - 7\rho + 2)y^2 + \\
(5\rho^6 - 3\rho^5 - 21\rho^4 + 18\rho^3 - \rho^2 - 3\rho + 1)xy + \\
\rho(\rho^5 + 8\rho^4 - 6\rho^3 - 5\rho^2 + 5\rho - 1)x^2 - \\
\rho(\rho-1)(2\rho^4 - 3\rho^3 - 7\rho^2 + 5\rho - 1)y - \\
\rho(\rho-1)(2\rho^4 + 7\rho^3 - 5\rho^2 - \rho + 1)x + \\
\rho^2(\rho^2 + 2\rho - 1)(\rho-1)^2
\end{array}\right|}{\left(\begin{array}{c}
(y + (\rho^2 - \rho + 1)x + \rho - 1)(\rho y + x + \rho(\rho-1)) \times \\
(\rho^2 y + \rho x - \rho + 1) \times \\
((\rho^2 - \rho + 1)y + \rho^2 x - \rho(\rho-1))
\end{array}\right)}.
$$

An affine image of a convex figure is convex. Convexity means that (x, y) is constrained to $\mathfrak{R}$, the open 'north-east corner' of the first quadrant bounded by $Y \cup T \cup X$, $Y = \{(0, y) : y \geqslant 1\}$, $T = \{(x, 1-x) : 0 \leqslant x \leqslant 1\}$, $X = \{(x, 0) : x \geqslant 1\}$. The image of S^* is the union of two mutually perpendicular linear sets

$$K = \left\{(x, y) : y - 1 = \rho(x-1), x > \frac{\rho}{\rho+1}\right\}$$

and

$$L = \left\{(x, y) : y - 1 = -\frac{1}{\rho}(x-1),\ 0 < x < 1 + \rho\right\}.$$

So we must show that $r(x, y) = M$ when $(x, y) \in K \cup L$, and $m < r(x, y) < M$ for all other (x, y) in $\mathfrak{R}$. We first study r on the boundary of $\mathfrak{R}$ and then use the boundary behaviour to help determine the interior behaviour. Restricting r to Y, we get a formula for $r(y) = r(0, y)$ which allows us to compute that at the endpoints of Y, $r(1) = r(\infty) = m$. (By $r(\infty) = m$, we mean $\lim_{y\to\infty} r(0, y) = m$.) Also, $r\left(1 + \frac{1}{\rho}\right) = M$. Taking the derivative gives the unexpectedly simple, completely factorised,

formula:

$$
\begin{aligned}
r'(y) &= \frac{\partial}{\partial y} r(0, y) \\
&= \frac{\rho^4(1-\rho)^2(\rho^2+1)(\rho y + 1 - \rho)}{((\rho^2 - \rho + 1)y + \rho(1-\rho))^2(\rho^2 y + 1 - \rho)^2}\left(1 + \frac{1}{\rho} - y\right).
\end{aligned}
$$

Since every factor except for the last one is positive at $y = 1$, and hence at every point of Y, the sign of r' agrees with the sign of the last factor. Thus r is increasing on the interval $\left(1, 1 + \frac{1}{\rho}\right)$ and decreasing on $\left(1 + \frac{1}{\rho}, \infty\right)$. In other words, r is mound-shaped on Y, taking on its maximum value M at $y = 1 + \frac{1}{\rho}$ and decreasing towards m as the endpoints of Y are approached.

We repeat this reasoning on T. Here we study $r(x) = r(x, 1-x)$, $0 \leqslant x \leqslant 1$. We have already calculated that $r(0) = m$ and a similar calculation shows that $r(1) = m$ also. We further compute that $r\left(\frac{\rho}{1+\rho}\right) = M$. To get r to be mound-shaped on T, we must show r' is positive on the interval $\left(0, \frac{\rho}{1+\rho}\right)$ and negative on $\left(\frac{\rho}{1+\rho}, 1\right)$. This is again clear since all factors except the last are positive when $0 \leqslant x \leqslant 1$ in the formula

$$
\begin{aligned}
r'(x) &= \frac{\partial}{\partial x} r(x, 1-x) \\
&= \frac{\rho^3(1-\rho)^2(1+\rho^2)(1+\rho)((1-\rho)x + \rho)}{(\rho(1-\rho)x + (\rho^2 - \rho + 1))^2((1-\rho)x + \rho^2)^2}\left(\frac{\rho}{1+\rho} - x\right).
\end{aligned}
$$

Similarly on X we study $r(x) = r(x, 0)$ getting $r(1) = r(\infty) = m$, $r(1+\rho) = M$, and

$$
\begin{aligned}
r'(x) &= \frac{\partial}{\partial x} r(x, 1-x) \\
&= \frac{\rho^3(1-\rho)^2(1+\rho^2)(x - 1 + \rho)}{((\rho^2 - \rho + 1)x + \rho - 1)^2(\rho^2 - \rho + x)^2}(1 + \rho - x)
\end{aligned}
$$

where all factors except the last are positive at $x = 1$ and hence on all of X. Once again, r is mound-shaped on X with edge values m and maximum value M.

Motivated by these results, we now sweep $\mathfrak{R}$ with line segments L_η with y-intercept η and slope $-\frac{1}{\rho}$. Differentiating r restricted to L_η gives

$$
\begin{aligned}
r'(x) &= \frac{\partial}{\partial x} r\left(x, -\frac{1}{\rho}x + \eta\right) \\
&= \frac{\rho^6(\rho-1)^2(\rho^2+1)^2\left(\eta - \frac{1+\rho}{\rho}\right)^2\left((\rho(\eta\rho + 1 - \rho)(\eta - 1 + \rho)) - ((1-\rho)^3 + (\rho - \rho^3)\eta)x\right)\left(\frac{\rho^2 + (\eta-1)\rho}{1+\rho^2} - x\right)}{\left(\eta\rho^2 + 1 - \rho\right)\left(\eta - 1 + \rho\right)\left(\eta\rho + (\rho^2 - \rho^3)(1-\eta) - (\rho^2+1)(1-\rho)x\right)^2\left(\rho(\eta - 1 + \rho) - x(\rho^2+1)(1-\rho)\right)^2}.
\end{aligned}
$$

If $\eta = \frac{1+\rho}{\rho}$, then $r' \equiv 0$; also L_η conuects the point of Y where $r = M$ to the point of X where $r = M$. This shows that r has constant value M on the linear set L. Also direct calculation shows that $r = M$ ou K. Henceforth we assume that $\eta \neq \frac{1+\rho}{\rho}$.

One endpoint of L_η is always in Y. We subdivide our proof into cases determined by the location of the other endpoint.

(1) If $\eta \geqslant \frac{1}{\rho}$, then one endpoint of L_η is at $(\eta\rho,\ 0) \in X$. Every factor of r' except the last one in the numerator is positive. (In particular, $f(x)$, the penultimate factor in the numerator, satisfies for $0 < x < \eta\rho$,

$$f(x) > f(\eta\rho) = \rho\left(\eta\rho^2(1-\rho)\right)\left(\rho + (\eta\rho - 1)\right) > 0.)$$

So r' is positive when $0 < x < \frac{\rho^2+(\eta-1)\rho}{1+\rho^2}$ and negative when $\frac{\rho^2+(\eta-1)\rho}{1+\rho^2} < x < \eta\rho$. Since L_η intersects K when $x = \frac{\rho^2+(\eta-1)\rho}{1+\rho^2}$, r is mound-shaped on L_η and the values of r are in (m, M) on $L_\eta \backslash K$.

(2) If $\frac{1}{\rho} > \eta > 1$, then one endpoint of L_η is on T and corresponds to $x = \frac{\rho(\eta-1)}{1-\rho}$. The argument is very similar to that in case (1). Here r is mound-shaped on L_η if $\frac{1}{\rho} > \eta > \frac{2}{1+\rho}$; but r is increasing on L_η if $\frac{2}{1+\rho} \geqslant \eta > 1$, because L_η leaves $\mathfrak{R}$ before intersecting K. The only delicate matter is the positivity of $f(x)$. But for $0 < x < \frac{\rho(\eta-1)}{1-\rho}, f(x) > f\left(\frac{\rho(\eta-1)}{1-\rho}\right) = \rho^2\eta\left(\rho + (1-\eta\rho)\right) > 0$.

To prove that the inner quadrilateral is a trapezium exactly when the ratio is maximal, it is enough to show that it being a trapezium is equivalent to (x, y) being on the image of S^*. Actually, in this case, the parallel sides are both parallel to the arm of S^* containing D. In view of the argument already presented, this is straightforward and will be left to the reader.

Recall that we have defined ρ-points in terms of counterclockwise orientation. Although Theorem 2 is true for clockwise orientation, we stress that the value of r depends, in general, on the orientation. In fact, clockwise and counterclockwise orientations always give different values of r unless D lies on the diagonal $\overrightarrow{BP}$.

Setting $\rho = \frac{1}{2}$ in Theorem 2 yields the following corollary.

Corollary 3: Let A, B, C, D be (counterclockwise) successive vertices of a convex quadrilateral. Define $EFGH$ as the inner quadrilateral formed by joining vertices to midpoints as described in Section 1. Construct the point P so that $ABCP$ is a parallelogram. Let Q be the midpoint of $\overline{AB}$, locate R on $\overrightarrow{BC}$ so that R is a distance $\frac{1}{2}BC$ from C and C is between B and R, and let $S = \overrightarrow{QP} \cup \overrightarrow{RP}$. Then the ratio r defined by (1) is maximal exactly when D is on $S^* = S \cap \operatorname{int}(\angle ABC) \cap \operatorname{ext}(\Delta ABC)$. The set of possible ratios is

$$\left(\frac{1}{6}, \frac{1}{5}\right].$$

Also the interior quadrilateral $EFGH$ is a trapezium (with at least two parallel sides) if and only if the ratio is $\frac{1}{5}$.

In a personal communication, Michael de Villiers told us that this Corollary was first proved by Coleman, Eberhart and Sathaye in the unpublished paper [6]. The idea of characterising the extremal case in terms of the interior quadrilateral being a trapezium is due to the authors of [6]. We arrived at our other conclusions independently.

Another corollary of Theorem 2 is the following generalisation of Theorem 1 from midpoints to ρ-points.

Corollary 2: If each vertex of a parallelogram is joined to the ρ-point of an opposite side in counterclockwise order to form an inner quadrilateral, then the area of the inner quadrilateral is $\frac{(1-\rho)^2}{\rho^2+1}$ times the area of the original parallelogram.

A nice geometry exercise is to prove this corollary by avoiding the calculus part of the proof of Theorem 2. Hint: Performing the affine transformation we may assume that the original quadrilateral is the unit square. Use slope considerations to see that the interior quadrilateral is actually a rectangle. Use length considerations to see that it is a square of side length $\sqrt{\frac{(1-\rho)^2}{\rho^2+1}}$.

4. *The non-convex case*

Nothing as tidy as Theorem 2 can happen here. The derived quadrilateral may not be inside the original quadrilateral; in fact the area of the derived quadrilateral may far exceed the original area. Also the derived quadrilateral may be 'twisted', that is, self-intersecting. In this case, it is probably most natural to take the area to be the signed area which is automatically generated by the quadrilateral area formula we use; this amounts to the difference between the areas of the two triangles formed. This area may even be zero, or a negative number.

We can illustrate all of these phenomena by restricting to the midpoint case, $\rho = \frac{1}{2}$. We further assume the same normalisation used in the proof of Theorem 2, so that $A = (0, 1)$, $B = (0, 0)$, $C = (1, 0)$ and $D = (x, y)$. If we look at the non-convex cases generated by letting $D \in \left\{(x, x) : 0 < x < \frac{1}{2}\right\}$, then the formula for the ratio becomes

$$r(x, x) = \frac{99x^4 + 54x^3 - 31x^2 + 4x - 1}{(7x-2)(6x-1)(4x+1)(3x+2)}, \qquad 0 < x < \frac{1}{2}.$$

If $x = 0.3261\ldots$ is the zero of the numerator lying between 0 and $\frac{1}{2}$, then the area of the derived quadrilateral is zero. (Geometer's Sketchpad confirms that the derived quadrilateral has the expected 'bow tie' shape, although the two triangles of equal area are not congruent.) Furthermore, graphing $r(x, x)$ shows that the ratio may achieve every value in the set $\left(-\infty, \frac{1}{5}\right] \cup [1, \infty)$.

There is room for further investigation here. For example,

(1) can the ratio r achieve values in $\left(\frac{1}{5}, 1\right)$ when $\rho = \frac{1}{2}$,

(2) what happens for other $\rho \in (0, 1)$, and

(3) what happens if the original quadrilateral is itself twisted?

Acknowledgement

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Student Problem Corner Editor

Tim Cross, who has edited the Student Problem Corner for 17 years, has indicated that he thinks it is now time to give up the role. It involves setting the problems, ensuring that they are of an appropriate standard to challenge school pupils, and then assessing the solutions which are received. Any reader of the *Gazette* who is interested in taking over is invited to contact the Editor to receive further information about what is involved.

Teaching Notes

Symmetric functions in the classroom

1. *Introduction*

In the Further Pure 1 module of the OCR A level syllabus, students are given a gentle introduction to the idea of *symmetric rational functions* (SRFs). An SRF in the variables $x_1, x_2, \ldots, x_n$ is left unchanged by any permutation of these variables (unchanged, that is, other than in the order of the terms and factors). To take an example,

$$\frac{x_1^2x_2}{x_3} + \frac{x_1x_2^2}{x_3} + \frac{x_1^2x_3}{x_2} + \frac{x_1x_3^2}{x_2} + \frac{x_2^2x_3}{x_1} + \frac{x_2x_3^2}{x_1}$$

is a SRF in x_1, x_2 and x_3. The *elementary symmetric polynomial* (ESP) $e_{k,n}$ is defined as the sum of all possible products of k distinct elements from $x_1, x_2, \ldots, x_n$. So the ESPs in three variables are given by $e_{1,3} = x_1 + x_2 + x_3$, $e_{2,3} = x_1x_2 + x_2x_3 + x_1x_3$ and $e_{3,3} = x_1x_2x_3$.

A well-known result is that any SRF in n variables can always be expressed as a rational fuction in $e_{1,n}, e_{2,n}, \ldots, e_{n,n}$. We may therefore think of ESPs as basic building blocks for SRFs. Students are expected to use this result iu order to deal with questions such as:

> If α and β are the roots of the equation $x^2 + 6x + 11 = 0$ then, without explicitly finding α and β, evaluate
> $$\frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha}.$$

From the given quadratic equation we see that $\alpha + \beta = -6$ and $\alpha\beta = 11$. Now $\alpha + \beta$ and $\alpha\beta$ are the ESPs in α and β, so we may arrange the above expression in terms of these polynomials and hence evaluate it without having first to find α and β themselves.

We have found that A level textbooks tend to gloss over this result, and the purpose of the present note is to encourage teachers of Further Mathematics to redress the balance in this regard. While a proof of the general case might be asking a little too much of many sixth-form classes, the special case of SRFs in two variables would certainly be accessible to any reasonably bright student. Indeed, the first-named author, who is a sixth-form student, is essentially responsible for the proof given here for this special case.

2. *The case* $n = 2$

Let us consider some SRF in α and β, $r(\alpha, \beta)$ say. We may write

$$r(\alpha, \beta) = \frac{f(\alpha, \beta)}{g(\alpha, \beta)},$$

where $f(\alpha, \beta)$ and $g(\alpha, \beta)$ are symmetric polynomials (SPs). It therefore suffices to prove the result for SPs. For any non-constant SP, $f(\alpha, \beta)$ say,