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ON HYPERBOLIC MEASURES AND PERIODIC ORBITS

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Dedicated to Anatole Katok on the occasion of his 60th birthday

Abstract. We prove that if a diffeomorphism on a compact manifold preserves a nonatomic hyperbolic Borel probability measure, then there exists a hyperbolic periodic point such that the closure of its unstable manifold has positive measure. Moreover, the support of the measure is contained in the closure of all such hyperbolic periodic points. We also show that if an ergodic hyperbolic probability measure does not locally maximize entropy in the space of invariant ergodic hyperbolic measures, then there exist hyperbolic periodic points that satisfy a multiplicative asymptotic growth and are uniformly distributed with respect to this measure.

1. **Introduction.** The theory of nonuniformly hyperbolic dynamical systems, often called *Pesin theory* due to the fundamental work of Ya. Pesin in the mid-seventies ([16, 17]), studies smooth dynamical systems preserving a hyperbolic measure, i.e., a measure whose Lyapunov exponents are nonzero almost everywhere. In his seminal work [6], Anatole Katok established several essential results about the periodic orbits of nonuniformly hyperbolic dynamical systems. He proved the closing property for such systems, found that the asymptotic exponential growth of periodic points of a diffeomorphism preserving a hyperbolic probability measure is bounded from below by its measure-theoretic entropy, and proved the existence of periodic orbits with transversal homoclinic intersections (hence also the existence of uniform hyperbolic horseshoes). Later, Katok [7] and Katok-Mendoza [9] proved the shadowing property, the Spectral Decomposition Theorem, the existence of hyperbolic horseshoes with topological entropy approximating (from below) the measure-theoretic entropy of the system.

Consider $f : M \to M$ to be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a smooth compact Riemannian manifold M preserving a hyperbolic Borel probability measure. In [6], Katok also proved that if the measure is nonatomic, then its support is contained in the closure of the hyperbolic periodic points that have transverse homoclinic points. In the same spirit, we show the following.

Theorem 1. *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism defined on a compact Riemannian manifold M. If f preserves a nonatomic hyperbolic Borel probability measure, then there exists a hyperbolic periodic point such that the closure of its global unstable manifold has positive measure. Moreover, the support of the measure is contained in the closure of all such hyperbolic periodic points.*

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The second result of this paper concerns the asymptotic growth of periodic orbits and their limit distribution in the context of nonuniformly hyperbolic dynamics. Let us recall that, for the uniformly hyperbolic situation, corresponding results are well known. For example, the theorem for Axiom A diffeomorphisms ([3]) can be stated as: if f is a topologically mixing Axiom A diffeomorphism on a compact manifold, then there exists a unique f-invariant measure of maximal entropy, obtained as the limit distribution of periodic points; moreover, $P_n(f) \sim e^{nh_{\text{top}}(f)}$, where $P_n(f)$ denotes the number of periodic points of period n, and $h_{\text{top}}(f)$ is the topological entropy of f. For topologically mixing Axiom A flows $\phi = \{\phi^t\}_{t\in\mathbb{R}}$, a similar statement is true for the measure of maximal entropy ([4]), but now $P_t(\phi) \sim \dfrac{e^{th_{\text{top}}(\phi)}}{th_{\text{top}}(\phi)}$, where $P_t(\phi)$ denotes the number of periodic orbits of period $\leq t$. This precise estimate was first obtained for Anosov flows by G. Margulis in his thesis (published recently in [12]), and extended by W. Parry and M. Pollicott [15] for Axiom A flows. The two asymptotic estimates for $P_n(f)$ and $P_t(\phi)$ are called *multiplicative*. They are stronger than exponential growth rates.

Let f be a $C^{1+\alpha}$ diffeomorphism on a smooth compact Riemannian manifold, and denote by $\mathcal{M}(f)$ the set of all f-invariant Borel probability measures, by $\mathcal{M}_e(f)$ the set of all ergodic measures in $\mathcal{M}(f)$, and by $\mathcal{M}_{eh}(f)$ the set of all hyperbolic ergodic measures in $\mathcal{M}(f)$. The set $\mathcal{M}(f)$ is endowed with the weak topology. We say that a measure $\mu \in \mathcal{M}_{eh}(f)$ is not locally maximal in the set $\mathcal{M}_{eh}(f)$ if any neighborhood of μ in $\mathcal{M}_{eh}(f)$ contains a measure $\tilde{\mu}$ of greater measure-theoretic entropy, i.e., $h_{\tilde{\mu}}(f) > h_\mu(f)$. The result we present establishes the existence of periodic orbits that satisfy a multiplicative asymptotic growth (in relation to $e^{nh_\mu(f)}$) and are uniformly distributed with respect to such a measure $\mu \in \mathcal{M}_{eh}(f)$ which does not maximize entropy locally.

Theorem 2. *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact Riemannian manifold M. Suppose that μ is an f-invariant hyperbolic ergodic Borel probability measure with $h_\mu(f) > 0$. If μ is not a locally maximal ergodic measure in the space of f-invariant ergodic hyperbolic measures, then for any $r > 0$ and any finite collection of continuous functions $\varphi_1, \dots, \varphi_k \in C(M)$, there exist a sequence $m_n \to \infty$ and sets $\mathcal{P}_{m_n} = \mathcal{P}_{m_n}(r, \varphi_1, \dots, \varphi_k)$ of hyperbolic periodic points of period m_n such that*

(i) $\operatorname{card} \mathcal{P}_{m_n} \geq e^{m_n h_\mu(f)}$;

(ii) $$\left| \frac{1}{m_n} \sum_{j=0}^{m_n-1} \varphi_i(f^j(z)) - \int \varphi_i d\mu \right| < r \quad \text{for any } z \in \mathcal{P}_{m_n}\,,\ i = 1, \dots, k\,.$$

Remark 1. For two-dimensional manifolds, it is not necessary to assume that μ is a hyperbolic measure. The fact that $h_\mu(f) > 0$ and Ruelle's inequality [18] assure us that the measure is indeed hyperbolic. The theorem can be stated in this case as follows: *if μ is not a locally maximal ergodic measure in the space of f-invariant ergodic measures, then there exist multiplicatively many (in relation to $e^{nh_\mu(f)}$) hyperbolic periodic orbits equidistributed with respect to μ.*

There are obvious situations when ergodic hyperbolic measures that are not locally maximal exist. If one considers, for example, a topologically mixing Axiom A diffeomorphism, then any ergodic invariant measure other than the measure of maximal entropy will not be locally maximal. This follows from the existence of Markov partitions and the abundance of measures associated to them (see [8]). Another example can be constructed as a product between a topologically mixing Axiom A diffeomorphism and a diffeomorphism with positive topological entropy, defined on a compact surface. Indeed, let f_1 be a topologically mixing Axiom A diffeomorphism on a compact manifold and μ_1 an f_1-invariant Borel probability measure other than the measure of maximal entropy. Also, let f_2 be a diffeomorphism on a compact surface and μ_2 an f_2-invariant ergodic Borel probability measure such that $h_{\mu_2}(f) > 0$ (such measures exist and are hyperbolic, since $h_{\text{top}}(f_2) > 0$ and f_2 is a surface diffeomorphism). Then the measure $\mu_1 \times \mu_2$ is an $f_1 \times f_2$-invariant hyperbolic Borel probability measure which does not maximize entropy locally.

We mention that fundamental results have been obtained by G. Knieper ([10, 11]) on the study of limit distribution of periodic orbits and their asymptotic estimates for an important example of nonuniformly hyperbolic flows—the geodesic flow on compact rank 1 manifolds of nonpositive curvature. The most precise asymptotics for this class of flows have been recently obtained by R. Gunesch [5].

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2. **Preliminaries.** In this section, we briefly summarize several notions and results from the theory of nonuniformly hyperbolic dynamical systems. For more details, see [2], [9].

Lyapunov exponents and hyperbolic measures. Let M be an m-dimensional smooth Riemannian manifold and $f : M \to M$ a $C^{1+\alpha}$ diffeomorphism on M, $\alpha > 0$. Given $x \in M$ and $v \in T_xM \setminus \{0\}$, define the *Lyapunov exponent of v at x* by the formula

$$\lambda(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\| .$$

If x is fixed, then $\lambda(x, \cdot)$ takes finitely many values $\lambda_1(x) < \cdots < \lambda_{k(x)}(x)$, with $k(x) \le m$.

Let us recall that the Oseledec Multiplicative Ergodic Theorem [13] (see also [2] for a detailed discussion) gives the existence of a set $\Lambda \subset M$ (whose points are called Lyapunov regular) such that:

(i) Λ is f-invariant and has full measure with respect to any f-invariant Borel probability measure;

(ii) for every $x \in \Lambda$, the tangent space T_xM admits the decomposition

$$T_xM = \bigoplus_{i=1}^{k(x)} E_i(x),$$

where each linear subspace $E_i(x)$ depends measurably on x and is invariant under the differential D_xf, $D_xf(E_i(x)) = E_i(f(x))$;

(iii) if $v \in E_i(x) \setminus \{0\}$, then

$$\lim_{n\to\pm\infty} \frac{1}{n} \|D_x f^n v\| = \lambda_i(x).$$

A Borel f-invariant probability measure μ is called *hyperbolic* if the Lyapunov exponents $\lambda_i(x)$ are different from zero for μ-almost every $x \in M$ and all $i = 1, \ldots, k(x)$. The functions $x \mapsto \lambda_i(x)$ are measurable and f-invariant.

Nouniformly hyperbolic dynamics. Let $h(x)$ be the largest integer such that $\lambda_i(x) < 0$ for $1 \le i \le h(x)$, and set

$$E^s(x) = \bigoplus_{i=1}^{h(x)} E_i(x) \qquad E^u(x) = \bigoplus_{i=h(x)+1}^{k(x)} E_i(x)\,.$$

These spaces are called *stable* and *unstable*, respectively. Pesin proved that the set Λ has a nonuniformly hyperbolic structure (see [2] for more details). In particular, there exist local (global) stable and unstable manifolds at every point $x \in \Lambda$. We denote by $W^s(x)$ and $W^u(x)$ the global stable and unstable manifolds at x.

Regular Neighborhoods. For a fixed $\epsilon > 0$, there exists a measurable function $q : \Lambda \to (0, 1]$ with $e^{-\epsilon} < q(f(x))/q(x) < \epsilon$, and for each $x \in \Lambda$ one can find a collection of embeddings $\Psi_x : B^s(0, q(x)) \times B^u(0, q(x)) \subset \mathbb{R}^{s(x)} \times \mathbb{R}^{m-s(x)} \to M$ (where $s(x) = \dim E^s(x)$), such that $\Psi_x(0) = x$ and, if $f_x = \Psi^{-1}_{f(x)} \circ f \circ \Psi_x$, then:

(i) The derivative $D_0 f_x$ of f_x at 0 has the Lyapunov block form:

$$A_\epsilon(x) = \begin{pmatrix} A^1_\epsilon(x) & & \\ & \dots & \\ & & A^k_\epsilon(x) \end{pmatrix},$$

where for each $i = 1, \dots, k(x)$, $A^i_\epsilon(x)$ is an $k_i(x) \times k_i(x)$ matrix and

$$e^{\lambda_i(x)-\epsilon} \le \left\| A^i_\epsilon(x)^{-1} \right\|^{-1}, \left\| A^i_\epsilon(x) \right\| \le e^{\lambda_i(x)+\epsilon}\,;$$

(ii) The C^1 distance between f_x and $D_0 f_x$ is at most ϵ;

(iii) There exist a constant $K > 0$ and a measurable function $A : \Lambda \to \mathbb{R}$ such that for $y, z \in B^s(0, q(x)) \times B^u(0, q(x))$,

$$K d(\Psi_x(y), \Psi_x(z)) \le \|y - z\| \le A(x) d(\Psi_x(y), \Psi_x(z))\,,$$

with $e^{-\epsilon} < A(f(x))/A(x)$.

The set $R(x) = \Psi_x(B^s(0, q(x)) \times B^u(0, q(x))) \subset M$ is called a *regular neighborhood* of the point x.

Pesin sets. For any $\delta > 0$, there exist a (noninvariant) uniformly hyperbolic compact set Λ_δ (called *Pesin set*) and $\epsilon = \epsilon(\delta)$ such that $\mu(\Lambda_\delta) > 1 - \delta$ and the functions $x \mapsto q(x)$, $x \mapsto A_\epsilon(x)$ are continuous on Λ_δ. The splitting $T_x M = E^s(x) \oplus E^u(x)$ varies continuously on Λ_δ. Denote by $q_\delta = \min\{q(x) \mid x \in \Lambda_\delta\}$.

Admissible manifolds. We say that a set $W \subset M$ is an admissible (u, γ)-manifold near $x \in \Lambda_\delta$ if $W = \Psi_x(\text{graph}\phi)$, where ϕ is a C^1 map

$$\phi : B^u(0, q_\delta) \to B^s(0, q_\delta) \text{ with } \|\phi(0)\| \le q_\delta/4\,,\ \|D_0\phi\| \le \gamma\,.$$

Similarly, we say that W is an admissible (s, γ)-manifold near $x \in \Lambda_\delta$ if $W = \Psi_x(\text{graph}\phi)$, where $\phi \in C^1(B^s(0, q_\delta), B^u(0, q_\delta))$ with $\|\phi(0)\| \le q_\delta/4$, $\|D_0\phi\| \le \gamma$. We have the following two properties for admissible manifolds ([9]):

(M1) There exists $\gamma = \gamma(\delta)$ such that any admissible (u, γ)-manifold near $x \in \Lambda_\delta$ intersects any admissible (s, γ)-manifold near x at exactly one point, and the intersection is transverse.

(M2) There exists ρ_δ such that if $x, y \in \Lambda_\delta$, $d(x, y) < \rho_\delta$ and W is an admissible (u, γ)-manifold near y, then W is an admissible (u, γ)-manifold near x.

In what follows, we restrict our discussion to the case when μ is an ergodic hyperbolic Borel probability measure. In this context, the functions $\lambda_i(x)$, $k_i(x)$, $k(x)$ and $s(x)$ (which are measurable and f-invariant) are constant μ-almost everywhere.

An essential tool for detecting periodic orbits in hyperbolic dynamics is the Closing Lemma. Katok ([6]) obtained this result for the nonuniformly hyperbolic situation.

Lemma 1 (The Closing Lemma ([6]))**.** *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism preserving a hyperbolic Borel probability measure. For all $\delta > 0$ and $\epsilon > 0$, there exists $\beta = \beta(\delta, \epsilon) > 0$ such that if $x, f^{n(x)}(x) \in \Lambda_\delta$ (for some $n(x) > 0$), and $d(x, f^{(n(x)}(x)) < \beta$, then there exists a hyperbolic periodic point $z(x)$ of period $n(x)$ with $d(f^k(z), f^k(x)) < \epsilon$ for every $k = 0, \dots, n(x) - 1$. Moreover, the stable and unstable manifolds of $z(x)$ are admissible $(s, \gamma(\delta))$- and $(u, \gamma(\delta))$-manifolds, respectively.*

Let us remark that the above result implies that the support of measure μ is contained in the closure of the set of all hyperbolic periodic points.

We end this section by recalling the statement about the existence of uniformly hyperbolic horseshoes with topological entropy arbitrarily close to $h_\mu(f)$.

Theorem 3 ([9])**.** *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism preserving an ergodic hyperbolic Borel probability measure μ, with $h_\mu(f) > 0$. Then for any $\rho > 0$ and any finite collection of continuous functions $\varphi_1, \dots, \varphi_k \in C(M)$, there exists a hyperbolic horseshoe Γ such that*

(i) $h_\mu(f) - \rho < h_{top}(f_{|\Gamma})$;
(ii) *Γ is contained in a ρ neighborhood of suppμ;*
(iii) *There exists a measure $\tilde\mu = \tilde\mu(\Gamma)$ supported on Γ such that for $i = 1, \dots, k$,*

$$\Big| \int \varphi_i d\tilde\mu - \int \varphi d\mu \Big| < \rho \, .$$

3. **Proofs.**

Proof of Theorem 1. Let $x \in \operatorname{supp}\mu$ and let $\eta > 0$ be a small enough. There exists $\delta > 0$ such that $\mu(B(x, \eta/2) \cap \Lambda_\delta) > 0$. Now let $r > 0$ be an arbitrary small number, with $r \le \min(\eta/2, q_\delta, \rho_\delta)$ (q_δ and ρ_δ are from the definition and property (M2) of admissible manifolds).

Pick a set $B \subset B(x, \eta/2) \cap \Lambda_\delta$ of diameter less than $\beta = \beta(\delta, r)$ (as in the Closing Lemma), less than r and of positive measure. Let $x_1 \in B$ be a recurrent point (by the Poincaré Recurrence Theorem), and $n(x_1)$ a positive integer such that $f^{n(x_1)}(x_1) \in B$. Since $d(x_1, f^{n(x_1)}(x_1)) < \beta$, in applying the Closing Lemma we obtain that there exists a periodic point z_1 of period $n(x_1)$ such that $d(x_1, z_1) < r$. This implies that $d(x, z_1) < \eta$.

We will prove that μ-almost every point of B belongs to $\overline{W^u(z_1)}$. For that, let $y \in B$ be a Borel density point, and let $\tau > 0$ be small enough such that $\tau < r/2$, $d(z_1, y) > \tau$ and $\mu(B \cap B(y, \tau/2)) > 0$. Pick a set $\tilde B \subset B \cap B(y, \tau/2)$ of diameter less than $\tilde\beta = \beta(\delta, \tau/2)$ and of positive measure. Let $x_2 \in \tilde B$ be a recurrent point, and $n(x_2)$ a positive integer such that $f^{n(x_2)}(x_2) \in \tilde B$. Since $d(x_2, f^{n(x_2)}(x_2)) < \tilde\beta$, in applying the Closing Lemma we obtain that there exists a periodic point z_2 of period $n(x_2)$ such that $d(x_2, z_2) < \tau/2$. This implies that $d(y, z_2) < \tau$, thus $z_1 \neq z_2$.

Since locally $W^s(z_2)$ is an admissible $(s, \gamma(\delta))$-manifold near x_2, and $d(x_2, x_1) \le \operatorname{diam}(B) < r \le \rho_\delta$, property (M2) of an admissible manifold implies that locally

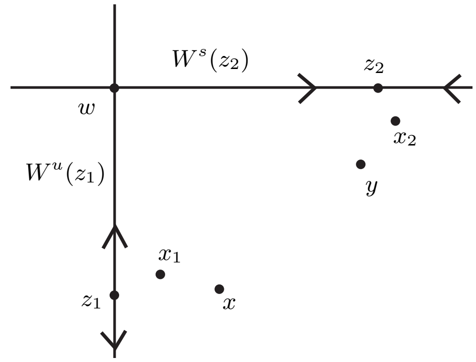

FIGURE 1. Transversal intersection

$W^s(z_2)$ is an admissible $(s, \gamma(\delta))$-manifold near x_1. Hence, due to the fact that locally $W^u(z_1)$ is an admissible $(u, \gamma(\delta))$-manifold near x_1, we obtain from property (M1) that $W^u(z_1)$ intersects transversely $W^s(z_2)$ at a point w (see Figure 1).

By the Inclination Lemma (see e.g. [8]), we have that $W^u(z_1)$ accumulates locally on $W^u(z_2)$. Indeed, let N be a common period for z_1 and z_2. The points z_1 and z_2 are hyperbolic fixed points for f^N with w a transverse intersection point of $W^u(z_1)$ and $W^s(z_2)$. The images under f^N of a ball around w in $W^u(z_1) = f^N(W^u(z_1))$ accumulate on $W^u(z_2)$. Hence

$$d(W^u(z_1), y) \leq d(W^u(z_1), x_2) + d(x_2, y) \leq \tau$$

for all $\tau > 0$. This implies that $y \in \overline{W^u(z_1)}$, for all Borel density points $y \in B$. Therefore $\mu(\overline{W^u(z)}) \geq \mu(B) > 0$.

Since the hyperbolic periodic point z_1 is such that $d(x, z_1) < \eta$, where x is an arbitrary point of $\operatorname{supp} \mu$ and $\eta > 0$ is arbitrary small, we obtain that the support of the measure μ is contained in the closure of all hyperbolic periodic points z with $\mu(\overline{W^u(z)}) > 0$. □

Remark 2. In many numerical studies of dynamical systems attesting the presence of a chaotic attractor (see [1, 14] and the references therein), it has also been noticed that the attractor coincides with the closure of the unstable manifold of a fixed or periodic hyperbolic orbit. We have just proved that this must be the situation if the corresponding dynamical system preserves an absolutely continuous hyperbolic measure or, more generally, a hyperbolic SRB (Sinai-Ruelle-Bowen) measure.

Now, we proceed with the proof of Theorem 2.

Proof of Theorem 2. The underlying idea of the argument is the fact that, since the measure μ is not locally maximal, one can find (using Theorem 3) a hyperbolic horseshoe with topological entropy greater than $h_\mu(f)$ and arbitrarily close to $h_\mu(f)$. This provides the possibility of choosing multiplicatively many hyperbolic periodic points equidistributed with respect to μ. The proof we present is independent of Theorem 3, although some of the ideas are borrowed from the proof of that theorem (see [9]).

We begin by recalling the definition of measure-theoretic entropy using d_n^f metrics (as described in [6]). Let

$$d_n^f(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)),$$

where d is the Riemannian distance on M. For $\epsilon > 0$ and $\delta > 0$, let $N_\mu(f, n, \epsilon, \delta)$ be the minimal number of ϵ-balls $B_n^f(x, \epsilon)$ in the d_n^f-metric which cover a set of measure more than or equal to $1 - \delta$. One has ([6, Theorem 1.1])

$$h_\mu(f) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log N_\mu(f, n, \epsilon, \delta)}{n} = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log N_\mu(f, n, \epsilon, \delta)}{n} .$$

Now let $r > 0$ be arbitrary small and $\varphi_1, \dots, \varphi_k \in C(M)$. Since the hyperbolic measure μ is not locally maximal in the class $\mathcal{M}_{eh}(f)$ of ergodic hyperbolic measures, we can find a hyperbolic measure $\tilde{\mu}$ such that $h_{\tilde{\mu}}(f) - 3r > (1 + r)h_\mu(f)$ and

$$\left| \int \varphi_i d\tilde{\mu} - \int \varphi_i d\mu \right| < r/3 \ , \ i = 1, \dots, k.$$

Choose $\delta > 0$ and $\epsilon > 0$ such that

(1) $\limsup_{n \to \infty} \dfrac{\log N_{\tilde{\mu}}(f, n, \epsilon, 2\delta)}{n} > h_{\tilde{\mu}}(f) - r$, and
(2) if $d(x, y) < \epsilon$, then $|\varphi_i(x) - \varphi_i(y)| < r/3$.

Let ξ be a finite partition of M with $\operatorname{diam} \xi < \beta(\delta, \epsilon/3)$ (from the Closing Lemma) and $\xi > \{\Lambda_\delta, M \setminus \Lambda_\delta\}$. Consider the set

$$\begin{aligned} \Lambda_{\delta,n} \quad = \quad & \Big\{ x \in \Lambda_\delta : f^m(x) \in \xi(x) \text{ for some } m \in [n, (1+r)n] \\ & \text{and } \left| \frac{1}{s} \sum_{j=0}^{s-1} \varphi^i(f^j(x)) - \int \varphi_i d\tilde{\mu} \right| < \frac{r}{3} \ \text{ for } s \geq n, i = 1, \dots, k \Big\}. \end{aligned}$$

Using Birkhoff's ergodic theorem, one can prove (see [9]) that $\tilde{\mu}(\Lambda_{\delta,n}) \to \tilde{\mu}(\Lambda_\delta)$. Choose now n sufficiently large such that $\tilde{\mu}(\Lambda_{\delta,n}) > 1 - 2\delta$. Let $E_n \subset \Lambda_{\delta,n}$ be an (n, ϵ)-separated set of maximal cardinality. One has

$$\Lambda_{\delta,n} \subset \bigcup_{x \in E_n} B_n^f(x, \epsilon),$$

and using relation (1) from above, there exist infinitely many n such that

$$\operatorname{card} E_n \geq e^{n(h_{\tilde{\mu}}(f) - 2r)}.$$

By the Closing Lemma, for each $x \in E_n$, there exists a hyperbolic periodic point $z(x)$ of period $m(x)$ such that $d(f^k(z), f^k(x)) < \epsilon/3$ for every $k = 0, \dots, m(x) - 1$. If $x, y \in E_n$, then $z(x) \neq z(y)$. Denote by $\mathcal{P}_m$ the set of all such periodic points of fixed period $m \in [n, (1+r)n]$. We have

$$\sum_{m=n}^{(1+r)n} \operatorname{card} \mathcal{P}_m \geq \operatorname{card} E_n,$$

hence

$$\max_{n \leq m \leq (1+r)n} \operatorname{card} \mathcal{P}_m \geq \frac{\operatorname{card} E_n}{rn} .$$

Therefore, we can find a sequence m_n, $n \leq m_n \leq (1+r)n$ such that

$$\operatorname{card} \mathcal{P}_{m_n} \geq \frac{\operatorname{card} E_n}{rn} \geq \frac{e^{n(h_{\tilde{\mu}}(f) - 2r)}}{rn} = e^{n(h_{\tilde{\mu}}(f) - 3r)} \geq e^{n(1+r)h_\mu} .$$

This implies that

$$\operatorname{card} P_{m_n} \geq e^{m_n h_\mu(f)} .$$

Moreover, for $z(x) \in \mathcal{P}_{m_n}$, we have

$$\left|\frac{1}{m_n}\sum_{j=0}^{m_n-1}\varphi_i(f^j(z)) - \int \varphi_i d\mu\right| \le \left|\frac{1}{m_n}\sum_{j=0}^{m_n-1}\varphi_i(f^j(z)) - \frac{1}{m_n}\sum_{j=0}^{m_n-1}\varphi_i(f^j(x))\right|$$
$$+\left|\frac{1}{m_n}\sum_{j=0}^{m_n-1}\varphi_i(f^j(x)) - \int \varphi_i d\tilde{\mu}\right| + \left|\int \varphi_i d\tilde{\mu} - \int \varphi_i d\mu\right| < \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r\,.$$

This finishes the proof of the theorem. □

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