

A view on extending morphisms from ample divisors

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Abstract. The philosophy that “a projective manifold is more special than any of its smooth hyperplane sections” was one of the classical principles of projective geometry.

Lefschetz type results and related vanishing theorems were one of the typically used techniques.

We shall survey some of the problems, results and conjectures in this area, using the modern setting of ample divisors, and (some aspects of) Mori theory.

Let X be a smooth projective variety over the complex field \mathbb{C} .

Let $n := \dim(X) \geq 3$

(in fact, for simplicity we will assume $n \geq 4$ in the sequel).

Let Y be a smooth ample divisor on X (sometimes it is enough to assume Y normal).

PROBLEM: To determine (classify) X “knowing Y ”.

Note two differences with the *classical situation*, that is, when $Y \subset X \subset \mathbb{P}^r$ is a hyperplane section:

1. $\mathcal{O}_Y(Y) \cong N_{Y/X}$; $\mathcal{O}_X(1) \cong \mathcal{O}_{\mathbb{P}^r}(1)|_X$,
i.e., the normal bundle is “a priori given”;

2. “more freedom” in our setting (Y merely ample).

In practice, “knowing Y ” means that we are given a map $p : Y \rightarrow Z$ (we may further assume p has connected fibers and Z is normal).

We would like to extend p to a morphism $\bar{p} : X \rightarrow Z$, giving a commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ p \downarrow & \swarrow \bar{p} & \\ Z & & \end{array} \quad (1)$$

And in case this is not possible, to explain why.

Notation: $\dim(p) :=$ dimension general fiber of p .

§1. The classical approach—via Lefschetz type results.

A basic tool for dealing with the above Problem is Lefschetz's theorem, which, in turn, is very much related (in fact, almost equivalent) to vanishing results of Kodaira type (see C.P. Ramanujam '72 and Kollár '87). Using these techniques, Sommese obtained the following two basic results.

Theorem 1.1. (Sommese '75)

Assume $\dim(p) \geq 2$ (and Y normal). Then $p : Y \rightarrow Z$ extends to $\bar{p} : X \rightarrow Z$.

Theorem 1.2. (Sommese '75)

Assume $\dim(p) \geq 1$ (and Y normal). Further assume that p is everywhere of maximal rank. If $p : Y \rightarrow Z$ extends to $\bar{p} : X \rightarrow Z$, then

$$\dim(X) \geq 2 \dim(Z).$$

[Equivalently, $\dim(Z) \leq \dim(\bar{p}).$]

§2. Mori theory techniques

To go further, we remark that, when

$$\dim(p) \leq 1,$$

the Problem becomes harder, and, in all interesting cases, p turns out to be a Mori contraction (e.g., a conic bundle, in particular a \mathbb{P}^1 -bundle, a blowing-up, etc. ...). It is therefore natural to attack the problem using techniques coming from Mori theory.

Assume, for simplicity, $n \geq 4$. Then, Lefschetz theorem gives an isomorphism

$$i^* : \text{Pic}(X) \xrightarrow{\cong} \text{Pic}(Y),$$

induced by the inclusion $i : Y \hookrightarrow X$.

In general, ample divisors on Y do not come as restrictions of ample divisors on X . In order to understand this phenomenon one can use the Kleiman–Mori cone.

Note that i also induces an isomorphism of \mathbb{R} -vector spaces

$$\bar{i} : N_1(Y) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

which maps injectively the Kleiman–Mori cone $\overline{NE}(Y)$ into $\overline{NE}(X)$,

$$\bar{i} : \overline{NE}(Y) \hookrightarrow \overline{NE}(X).$$

Here $\overline{NE}(\cdot)$ denotes the closure in \mathbb{R}^ρ of the convex cone $NE(\cdot)$ generated by effective 1-cycles, where $\rho := \dim_{\mathbb{R}} N_1(\cdot)$.

The **basic idea**, which is a refinement of Lefschetz theory, is to compare the Kleiman–Mori cones of Y and X , via the above inclusion.

Hopefully, e.g., if the two cones are equal, we can extend to X all extremal contractions of Y .

This was shown to hold, in the case when both Y and X are Fano manifolds by Kollár and Wiśniewski in the early '90's.

The general case was treated by Ionescu '99 and further extended by Andreatta–Occhetta to the case when Y is a section of an ample vector bundle on X , having the “right dimension”.

To end the general presentation, we point out that, in case $n = 3$, complete results have been obtained by Bădescu, Campana–Flenner, Ionescu and Sommese. They show that any particular property of Y (i.e., a property which follows from the canonical bundle K_Y to be not nef; e.g., existence of (-1) -curves, fibrations, etc....) strongly reflects on X . \square

The basic facts from Mori's theory that are used are the following:

1. Mori's Cone theorem;
2. Kawamata–Reid–Shokurov Basepoint free theorem;
3. the inequality

$$\dim(F) + \dim(\text{Locus}(R)) \geq \ell(R) + n - 1,$$

where $\text{Locus}(R)$ is the locus of curves whose numerical classes are in R , $\ell(R)$ is the length of the extremal ray R , and F is (an irreducible component of) a fiber of the contraction cont_R of R (cf. Ionescu '86, Wiśniewski '91, Kollár '96).

§3. Comparing cones results

Unless otherwise specified, let us assume $n = \dim X \geq 4$.

The following example shows that, in general, an extremal ray of Y may not extend

$$\begin{array}{c} Y \cong \mathbb{P}^1 \times \mathbb{P}^{n-2} \subset X \\ \downarrow p \\ \mathbb{P}^{n-2} \end{array}$$

However, the other projection, $q : Y \rightarrow \mathbb{P}^1$, does (by Sommese's extension theorem).

Other explicit examples where the inclusion $\overline{NE}(Y) \subset \overline{NE}(X)$ is strict are due to Bădescu, Andreatta–Occhetta, Wiśniewski, Mella).

The following result gives necessary and sufficient conditions for a given extremal ray of Y to extend to X .

Theorem 3.1. (Ionescu '99) *Let $p = \text{cont}_R : Y \rightarrow Z$, Y smooth ample divisor in X . Then p extends (necessarily to the contraction of an extremal ray of X) if and only if:*

$$\exists D \in \text{Pic}(X), D \text{ nef}, D_Y^\perp \cap \overline{NE}(Y) = R.$$

From this result it follows (with some work) the “key-fact”:

- Let Y be a smooth ample divisor on X . Assume that K_Y is not nef. Then some extremal ray of Y always extends.

(In fact, R is an extremal ray on Y such that $\frac{(Y \cdot C)}{\ell(R)}$ reaches the minimum, where C minimal rational curve on Y , $R = \mathbb{R}_+[C]$.)

The following Corollary shows how one may apply such a result, when the ray of Y is unique (the uniqueness in the statement below follows from a Wiśniewski's construction, '91).

Corollary 3.2. *Let $p : Y \rightarrow Z$ be a \mathbb{P}^1 -bundle and assume K_Z is nef. Then p extends to a \mathbb{P}^2 -bundle $\bar{p} : X \rightarrow Z$. Moreover, there is a non-split exact sequence*

$$0 \rightarrow \mathcal{O}_Z \rightarrow \mathcal{E} \xrightarrow{\sigma} \mathcal{F} \rightarrow 0,$$

\mathcal{E}, \mathcal{F} ample vector bundles (of rank 3, respectively 2) on Z , such that

$$\begin{array}{ccc} Y = \mathbb{P}(\mathcal{F}) & \subset & X = \mathbb{P}(\mathcal{E}) \\ & \searrow p & \swarrow \bar{p} \\ & Z & \end{array}$$

via σ .

This is related to the following still open conjecture:

Conjecture (—, Sommese) *Let $p : Y \rightarrow Z$ be a \mathbb{P}^1 -bundle. Then p extends to a \mathbb{P}^2 -bundle $\bar{p} : X \rightarrow Z$ unless one of the following holds:*

1. *Either $X = \mathbb{P}^3$, $Y \in |\mathcal{O}(2)|$, or $X = Q^3 \subset \mathbb{P}^4$, $Y \in |\mathcal{O}(1)|$; or*
2. *$n \geq 3$, $Y \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$, $p : Y \rightarrow \mathbb{P}^{n-2}$ ($q : Y \rightarrow \mathbb{P}^1$ extends to a \mathbb{P}^{n-1} -bundle).*

The conjecture holds true in the following cases:

1. $\dim Z = 1$ (Bădescu, '80/81);
2. $\dim Z = 2$ (Sommese et al., several papers \approx '86–'88);
3. $\dim Z = 3$ and Y is very ample. (—, Fania, Sommese, '05).

Note. If $p : Y \rightarrow Z$ extends then $\dim Z \leq 2$ by Sommese Theorem 1.2. Then case 3) above rephrases as:

Y cannot be a very ample divisor in X unless $Z = \mathbb{P}^3$ and $Y = \mathbb{P}^1 \times \mathbb{P}^3$. \square

A further result related to the conjecture is:

Theorem 3.3. (A variant of a result by Occhetta, '06) *Assume $p = \text{cont}_R : Y \rightarrow Z$ is the contraction of a nef extremal ray R on Y , $R = \mathbb{R}_+[C]$. If there exists an ample $H \in \text{Pic}(X)$, such that $(H \cdot C) = 1$, then p extends.*

Finally, we mention the following result dealing with the case when Y is Fano.

Theorem 3.4 (Improvement of —, Fania, Sommese '02) *Let Y be a smooth ample divisor on X . If Y is a Fano manifold (of index $i(Y) \geq 3$) then either:*

1. *X is also Fano and $\overline{NE}(Y) = \overline{NE}(X)$ (“optimal situation”: Y, X both Fano, so that any morphism extends); or*
2. *X has an extremal ray “of high length”.
[In fact, $\ell \geq 2i + 1$. So that, e.g., if $i(Y) \geq \frac{n}{2}$, then $\ell \geq n + 1$ giving the trivial case $X \cong \mathbb{P}^n$.]*

In closing, we mention three possible generalizations.

1. (de Fernex–Lanteri, Lanteri–Maeda, Andreatta–Occhetta and others)
 $Y \subset X$ is a section of the appropriate expected dimension of an ample vector bundle on X .
2. (Bădescu, Campana–Flenner and others)
 Y divisor in X is only supposed to have ample normal bundle. Then a classical well-known result shows that we can get a birational map $\psi : X \rightarrow X'$, which is an isomorphism around Y , such that $\psi(Y) := Y' \subset X'$ is ample and X' is normal.
3. (Occhetta; —, de Fernex, Lanteri)
 $\text{cd}_X Y \geq 2$, Y has ample normal bundle and $\text{Pic}(Y) \cong \text{Pic}(X)$. Then study extensions of rationally connected fibrations $p : Y \rightarrow Z$.