MONGE–AMPÈRE EQUATIONS AND BELLMAN FUNCTIONS: 
THE DYADIC MAXIMAL OPERATOR

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Abstract. We find explicitly the Bellman function for the dyadic maximal operator on \( L^p \) as the solution of a Bellman PDE of Monge–Ampère type. This function has been previously found by A. Melas [M] in a different way, but it is our PDE-based approach that is of principal interest here. Clear and replicable, it holds promise as a unifying template for past and current Bellman function investigations.

1. Introduction

For a locally integrable function \( g \) on \( \mathbb{R}^n \) and a set \( E \subset \mathbb{R}^n \) with \( |E| \neq 0 \), let \( \langle g \rangle_E = \frac{1}{|E|} \int_E g \) be the average of \( g \) over \( E \). Let \( p > 1 \) and \( q > 1 \) be conjugate exponents, i.e. \( p^{-1} + q^{-1} = 1 \). Let \( \varphi \) be a nonnegative locally \( L^p \)-function on \( \mathbb{R}^n \). Fix a dyadic lattice \( D \) on \( \mathbb{R}^n \) and consider the dyadic maximal operator

\[
M \varphi(x) = \sup_{I \ni x; I \in D} \langle \varphi \rangle_I.
\]

Following F. Nazarov and S. Treil [NT], we define the Bellman function for \( M \varphi \)

\[
B(f, F, L) = \sup_{0 \leq \varphi \in L^p_{\text{loc}}(\mathbb{R}^n)} \left\{ \langle (M \varphi)^p \rangle_Q : \langle \varphi \rangle_Q = f; \langle \varphi^p \rangle_Q = F; \sup_{R \supset Q} \langle \varphi \rangle_R = L \right\}.
\]

Observe that \( B \) is independent of \( Q \) and well-defined on the domain

\[
\Omega = \{(f, F, L) : 0 < f \leq L; f^p \leq F\}.
\]

Finding \( B \) will, among other things, provide a sharp refinement of the Hardy–Littlewood–Doob maximal inequality

\[
\|M \varphi\|_p \leq q \|\varphi\|_p.
\]

In [NT], the authors show that \( B(f, F, L) \leq q^p F - pqfL^{p-1} + pL^p \), which implies (1.2). A. Melas in [M], using deep combinatorial properties of the operator \( M \) and without relying on the Bellman PDE, finds \( B \) explicitly. In contrast, we develop a boundary value problem of Monge–Ampère type that \( B \) must satisfy (assuming sufficient differentiability) and solve it, producing the function from [M]. Our approach has been used as the foundation of several recent Bellman function results. We first restrict our attention to the one-dimensional case and then show that the Bellman function does not depend on dimension.

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Let $Q$ be an interval and $Q_-, Q_+$ its left and right halves, respectively. Let $(f_\pm, F_\pm) = (f_{Q_\pm}, F_{Q_\pm})$, $(f, F) = ((f_-, F_-) + (f_+, F_+))/2$. Taking suprema in the identity

$$\langle (M\varphi)^p \rangle_Q = \frac{1}{2} \langle (M\varphi)^p \rangle_{Q_-} + \frac{1}{2} \langle (M\varphi)^p \rangle_{Q_+}$$

over all $\varphi$ with appropriate averages, we obtain

$$B(f, F, L) \geq \frac{1}{2} B(f_-, F_-, \max \{f_-, L \}) + \frac{1}{2} B(f_+, F_+, \max \{f_+, L \}).$$

Any function $B$ satisfying this pseudo-concavity property on $\Omega$ will be a majorant of the true Bellman function. The following theorem phrases this condition in a differential form.

**Theorem 2.1.** Let $z = (f, F)$. Assuming sufficient smoothness on the Bellman function $B$, condition (2.1) holds for all admissible triples if and only if

$$\det \left( \frac{\partial^2 B}{\partial z^2} \right) = 0, \; B_{ff} \leq 0, \; B_L \geq 0 \text{ on } \Omega; \; 2B_{fL} + B_{LL} \leq 0, \; B_L = 0 \text{ when } f = L.$$

### 3. Homogeneity, Boundary Value Problem, Solution

We reduce the order of the PDE in (2.2) by using the multiplicative homogeneity of $B : B(f, F, L) = L^p B(f/L, F/L, 1)$ defined $L^p G(x, y)$, where $x = f/L, y = F/L^p$. In addition, $F = f^p$ only for functions that are constant on $Q$, so $B(f, f^p, L) = L^p$, meaning $G(x, x^p) = 1$. Coupling this with the first and the last conditions in (2.2), we get a boundary value problem for $G$ on the domain $\{(x, y) | 0 < x \leq 1; x^p \leq y \}$:

$$G_{xx} G_{yy} = G_{xy}^2; \quad G(x, x^p) = 1; \quad pG(1, y) = G_x(1, y) + pyG_y(1, y).$$

We look for the solution of the Monge–Ampère equation (3.1) in the general parametric form

$$G(x, y) = tx + f(t)y + g(t); \quad x + f'(t)y + g'(t) = 0.$$

Fix a value of $t$, i.e., fix one of the straight-line trajectories in (3.2). Let $(u(t), u'(t))$ be the point where that trajectory intersects the lower boundary $y = x^p$. We have

$$G(u, u^p) = tu(t) + f(t)u'^p(t) + g(t) = 1; \quad u(t) + f'(t)u'^p(t) + g'(t) = 0.$$

Differentiating the first equation and using the second one, we get, after some algebra, $f = -t/(pu'^{p-1})$, $g = 1 - tu/q$. Assume now that the trajectory intersects the right boundary $x = 1$ at the point $(1, v(t))$. Then $G(1, v) = t + fv + g$. On the other hand, parametrization (3.2) implies $G_x = t$, $G_y = f(t)$ and so the second boundary condition in (3.1) becomes $G(1, v) = \frac{t}{v} + fv$. This gives $g = -t/q$, allowing us to express $t = q/(u-1)$. Simplifying, we obtain a complete solution of the form (3.2):

$$G(x, y) = \frac{y}{u^p}; \quad x - qu - \frac{1}{qu} \cdot y - \frac{1}{q} = 0.$$

In terms of the original variables, we get a Bellman function candidate near the boundary $f = L$:

$$B(f, F, L) = Fu^{-p} (f/L, F/L^p).$$
4. FROM THE CANDIDATE TO THE TRUE FUNCTION

4.1. \( B \geq B \). One can readily verify that the rest of conditions (2.2) are satisfied by the candidate (3.4). Therefore, property (2.1) holds and one can perform the Bellman induction: take any non-negative function \( \varphi \in L^p_\text{loc}(\mathbb{R}^n) \) and an interval \( Q_0 \in D \). For an interval \( Q \subset Q_0, Q \in D \), let \( X_Q = (f_Q, F_Q, L_Q) \) with \( f, F, \) and \( L \) defined as in (1.1). Then

\[
B(f_Q, F_Q, L_Q) \geq \frac{1}{2}B(X_{Q_0})_+ + \frac{1}{2}B(X_{Q_0})_-
\]

(4.1)

\[
\geq \frac{1}{|Q_0|} \sum_{Q \subset Q_0, |Q|=2^{-n}|Q_0|} |Q|B(X_Q) \geq \frac{1}{|Q_0|} \sum_{Q \subset Q_0, |Q|=2^{-n}|Q_0|} |Q|L_Q^p
\]

\[
= \frac{1}{|Q_0|} \sum_{Q \subset Q_0, |Q|=2^{-n}|Q_0|} |Q|\left(\sup_{R \supset Q} (M\varphi)_R \right) \rightarrow \langle (M\varphi)^p \rangle_{Q_0}, \text{ as } n \to \infty.
\]

Here we have used that \( B \geq L^p \). Taking supremum on the right over all \( \varphi \) with the above \( X_{Q_0} \), we get \( B \geq B \).

4.2. \( B \leq B \). To get the reverse inequality, we need to construct, for every point \( (f, F, L) \in \Omega \), a sequence of nonnegative functions on \( (0, 1) \) \{\( \varphi_n \)\}, so that

\[
\lim_{n \to \infty} \langle (M\varphi_n)^p \rangle_{(0, 1)} \geq B(f, F, L).
\]

To do this, we use the trajectories \( t = \text{const} \) of the Monge–Ampère equation from Section 3. In the original variables, this gives

\[
f = \frac{L}{q} + AF.
\]

On the boundary \( f = L \) going along these trajectories yields the extremal sequence

\[
\varphi_n(t) = \begin{cases} 
\alpha_n L & 0 < t < 2^{-n} \\
\varphi_n(2^k t - 1) & 2^{-k} < t < 2^{-k+1}, k = 2, ..., n \\
\beta_n \varphi_n(2 t - 1) & \frac{1}{2} < t < 1.
\end{cases}
\]

(4.3)

The definition is understood recursively, whereby the function is defined on a portion of \( (0, 1) \), then on the same portion of the remaining part, and so on. The numbers \( \alpha_n \) and \( \beta_n \) are chosen so that \( \langle \varphi_n \rangle_{(0, 1)} = L \) and \( \langle \varphi_n^p \rangle_{(0, 1)} = F \). This means

\[
\frac{1}{2^n} \alpha_n + \frac{1}{2^n} \beta_n = 1, \quad \frac{1}{2^n} \alpha_n^p + \frac{1}{2^n} \beta_n^p F = \left( \frac{1}{2^n} + \frac{1}{2} \right) \frac{F}{L^p}.
\]

One can show that \( \alpha_n M \varphi_n \geq \varphi_n \) and \( \alpha_n \to u(1, F/L^p) \) with \( u \) defined by (3.3). Therefore,

\[
\lim_{n \to \infty} \langle (M\varphi_n)^p \rangle_{(0, 1)} \geq \lim_{n \to \infty} \frac{1}{\alpha_n^p} \langle \varphi_n^p \rangle_{(0, 1)} = \lim_{n \to \infty} \frac{F}{\alpha_n^p} = Fu^{-p}(1, F/L^p) = B(L, F, L),
\]

which gives \( B(L, F, L) \geq B(L, F, L) \).

On the boundary \( F = f^p \) the situation is simple: here the only test functions are constants and so \( B(f, f^p, L) = B(f, f^p, L) = L^p \). Having constructed the extremal sequences on the two boundaries, we get the extremal sequence at any point \((f, F, L)\) with \( f > L/q \) as their weighted dyadic rearrangement built along the unique extremal trajectory of the form (4.2) passing through the point.

One observes, however, that trajectories (4.2) cannot be used with \( A < 0 \), since they then would intersect the “forbidden” boundary \( f = 0 \). (It is forbidden because, for a nonnegative function, \( f = 0 \) implies \( F = 0 \).) In fact, in the region \( 0 < f < L/q \), no trajectory can lean either to the left or to the right (the forbidden boundary to the left, the existing extremal
trajectory \( f = L/q \) to the right). We conclude two things: the trajectories are vertical in this region and the candidate (3.4) no longer works there. However, this is quickly rectified: If \( G(x, y) = ax(x) + b(x) \), then \( G(x, x^p) = 1 \) implies that \( G(x, y) = 1 + a(x)(y - x^p) \). Now \( G_{xx}G_{yy} - G_{xy}^2 = -(a'(x))^2 = 0 \), and \( G(1/q, y) = q^p y \) implies that \( a(x) = q^p \). Thus we get the unique two-piece Bellman function candidate

\[
B(f, F, L) = \begin{cases} 
F u^{-p}(f/L, F/L^p) & L < qf \\
L^p + q^p(F - f^p) & L \geq qf.
\end{cases}
\]  

(In the notation of [M], \( u^{-p}(x, y) = \omega_p((px - p + 1)/y)^p \).) This \( B \) still satisfies (2.1). Therefore, Bellman induction (4.1) works. We now need an extremal sequence proving that \( B \geq B \) in the region \( L \geq qf \). There is a unique extremal trajectory passing through each point of the region. However, the trajectory is vertical and so intersects the boundary of \( \Omega \) at a single point; as a result we cannot use a weighted average of boundary extremal sequences like we just did for the region \( L > f/q \). We deal with it by tilting the trajectory slightly to the right, which produces a (distant) second boundary point, at the boundary \( f = L \). This lets us use the extremal sequence \( \varphi_n \) from (4.3), while simultaneously reducing the tilt. Namely, fix \( (f, F, L) \) and \( k \geq 1 \). Define \( \gamma_k \) and \( F_k \) so that \( L - \gamma_k = 2^k(f - \gamma_k) \) and \( F_k - \gamma_k = 2^k(F - \gamma_k) \). (Observe that \( \gamma_k \rightarrow f \) and \( F_k \rightarrow \infty \).) Using (4.3), form a sequence \( \{\varphi_{k,n}\}_{n=1}^{\infty} \) with \( \langle \varphi_{k,n} \rangle_{(0,1)} = L \) and \( \langle \varphi_{k,n} \rangle_{(0,2)} = F_k \), so that \( \langle (M\varphi_{k,n})^p \rangle_{(0,1)} \rightarrow B(L, F_k, L) \), as \( n \rightarrow \infty \). Let

\[
\psi_{k,n}(t) = \begin{cases} 
\varphi_{k,n}(2^kt) & 0 < t < 2^{-k} \\
\gamma_k & 2^{-k} < t < 1 \\
2L - f & 1 < t < 2.
\end{cases}
\]

Direct computation shows that \( \langle \psi_{k,n} \rangle_{(0,1)} = f \), \( \langle \psi_{k,n} \rangle_{(0,2)} = F \), and \( \langle \psi_{k,n} \rangle_{(0,2)} = L \). Then

\[
\langle (M\psi_{k,n})^p \rangle_{(0,1)} \geq L^p(1 - 2^{-k}) + 2^{-k}\langle (M\varphi_{k,n})^p \rangle_{(0,1)} \xrightarrow{n \rightarrow \infty} L^p(1 - 2^{-k}) + 2^{-k}B(L, F_k, L) \xrightarrow{k \rightarrow \infty} L^p + (F - F^p)u^{-p}(1, \infty) = L^p + q^p(F - f^p),
\]

5. Several dimensions

It turns out that the Bellman function (1.1),(4.4) is dimension-free. Fix a dyadic cube \( Q \) and let \( Q_1, \ldots, Q_{2^n} \) be its dyadic offspring. Then

\[
B \left( 2^{-n} \sum_{k=1}^{2^n} z_k, L \right) \geq 2^{-n} \sum_{k=1}^{2^n} B(z_k, \max\{f_k, L\}).
\]

Therefore, we can run the induction (4.1) to prove that \( B \geq B \). The other direction is shown by a trivial modification of the one-dimensional maximizing sequences. A similar argument can be used to show that the same Bellman function works for the maximal operator on trees, the setting of choice in [M].

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