A NOTE ON THE GUROV-RESHETNYAK CONDITION

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Abstract. An equivalence between the Gurov-Reshetnyak GR(ε) and Muckenhoupt A∞ conditions is established. Our proof is extremely simple and works for arbitrary absolutely continuous measures.

Throughout the paper, μ will be a positive measure on \( \mathbb{R}^n \) absolutely continuous with respect to Lebesgue measure. Denote

\[ \Omega_\mu(f; Q) = \frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}|d\mu(x), \quad f_{Q,\mu} = \frac{1}{\mu(Q)} \int_Q f(x)d\mu(x). \]

**Definition 1.** We say that a nonnegative function \( f, \mu \)-integrable on a cube \( Q_0 \), satisfies the Gurov-Reshetnyak condition \( GR_\mu(\varepsilon) \), \( 0 < \varepsilon < 2 \), if for any cube \( Q \subset Q_0 \),

\[ (1) \quad \Omega_\mu(f; Q) \leq \varepsilon f_{Q,\mu}. \]

When \( \mu \) is Lebesgue measure we drop the subscript \( \mu \).

This condition appeared in [6, 7]. It is important in Quasi-Conformal Mappings, PDEs, Reverse Hölder Inequality Theory, etc. (see, e.g., [2, 8]). Since (1) trivially holds for all positive \( f \in L_{\mu}(Q_0) \) if \( \varepsilon = 2 \), only the case \( 0 < \varepsilon < 2 \) is of interest. It was established in [2, 6, 7, 8, 13] for Lebesgue measure and in [4, 5] for doubling measures that if \( \varepsilon \) is small enough, namely \( 0 < \varepsilon < c2^{-n} \), the \( GR_\mu(\varepsilon) \) implies \( f \in L_p^\mu(Q_0) \) with some \( p > 1 \) depending on \( \varepsilon \). The machinery used in the articles mentioned above does not work for \( \varepsilon > 1/8 \) even in the one-dimensional case.

The one-dimensional improvement of these results was done in [9]. Namely, for any \( 0 < \varepsilon < 2 \) it was proved that \( GR(\varepsilon) \subset L_p^\mu \) where \( 1 < p < p(\varepsilon) \); moreover a sharp bound \( p(\varepsilon) \) for the exponent was discovered. The main tool in [9] is the Riesz Sunrising Lemma which has no multidimensional version since it involves the structure of open sets on a real line.

In the present article using simple arguments we prove that for any \( n \geq 1, 0 < \varepsilon < 2 \) and arbitrary absolutely continuous measure \( \mu \), the Gurov-Reshetnyak condition \( GR_\mu(\varepsilon) \) implies the weighted \( A_\infty(\mu) \) Muckenhoupt condition. And conversely, \( A_\infty(\mu) \) implies \( GR_\mu(\varepsilon_0) \) for some \( 0 < \varepsilon_0 < 2 \).

In the non-weighted (or doubling) case R.R. Coifman and C. Fefferman [3] have found several equivalent descriptions of the \( A_\infty \) property. Recently these descriptions have been transfered by J. Orobitg and C. Perez [12] to the non-doubling case. For our purposes it will be convenient to define \( A_\infty(\mu) \) by the following way.

**Definition 2.** We say that a nonnegative function \( f, \mu \)-integrable on a cube \( Q_0 \), satisfies Muckenhoupt condition \( A_\infty(\mu) \) if there exist \( 0 < \alpha, \beta < 1 \) such that for any cube \( Q \subset Q_0 \),

\[ \mu\{x \in Q : f(x) > \beta f_{Q,\mu}\} > \alpha \mu(Q). \]

Our main result is the following

1991 Mathematics Subject Classification. Primary 42B25.

The first author was partially supported by Ukrainian Foundation of Fundamental Research, grant F7/329 - 2001.
Corollary. The following characterization of $A_\infty(\mu)$ holds:

$$A_\infty(\mu) = \bigcup_{0 < \varepsilon < 2} GR_\mu(\varepsilon).$$

Since $A_\infty(\mu)$ condition is equivalent to the weighted reverse Hölder inequality for some $p > 1$ (cf. [12]), i.e.

$$\left( \frac{1}{\mu(Q)} \int_Q (f(x))^p d\mu(x) \right)^{1/p} \leq c \frac{1}{\mu(Q)} \int_Q f(x) d\mu(x) \quad (Q \subset Q_0), \quad (2)$$
we see that for any $0 < \varepsilon < 2$ a function $f$ satisfying the Gurov-Reshetnyak condition $GR_\mu(\varepsilon)$ belongs to $L^p(X)$ for some $p > 1$. Observe that such approach (i.e. $GR_\mu(\varepsilon) \Rightarrow A_\infty(\mu) \Rightarrow$ Reverse Hölder) does not give the optimal order of integrability for small $\varepsilon$, though it is known [2, 4] in doubling case that $GR_\mu(\varepsilon) \subset L^p_\mu(Q_0)$, where $p(\varepsilon) \sim c_n/\varepsilon, \varepsilon \to 0$, and this order is sharp. However we will show that part (i) of Theorem 1 allows us to obtain the same order for any measure $\mu$, any $0 < \varepsilon < 2$, and $f \in GR_\mu(\varepsilon)$. We will need the following

**Covering Lemma** [10]. Let $E$ be a subset of $Q_0$, and suppose that $\mu(E) \leq \rho \mu(Q_0), \ 0 < \rho < 1$. Then there exists a sequence $\{Q_i\}$ of cubes contained in $Q_0$ such that

(i) $\mu(Q_i \cap E) = \rho \mu(Q_i)$;
(ii) the family $\{Q_i\}$ is almost disjoint with constant $B(n)$, that is, every point of $Q_0$ belongs to at most $B(n)$ cubes $Q_i$;
(iii) $E' \subset \bigcup_j Q_j$, where $E'$ is the set of $\mu$-density points of $E$.

Recall that the non-increasing rearrangement of $f$ on a cube $Q_0$ with respect to $\mu$ is defined by

$$f^{**}_\mu(t) = \sup_{E \subset Q_0: \mu(E) = t} \inf_{x \in E} |f(x)| \quad (0 < t < \mu(Q_0)).$$

Denote $f^{**}_\mu(t) = t^{-1} \int_0^t f^{*}_\mu(\tau) d\tau$.

**Theorem 2.** Let $0 < \varepsilon < 2$, and $f \in GR_\mu(\varepsilon)$. Then for $\varepsilon < \lambda < 2$, $\rho < 1 - \lambda/2$, and $t \leq \rho \mu(Q_0)$ we have

$$f^{**}_\mu(t) \leq \left( B(n) \frac{\lambda/\rho + 1}{\lambda - \varepsilon} + 1 \right) f^{*}_\mu(t).$$

**Remark.** A well-known argument due to Muckenhoupt (see, e.g., [11, Lemma 4]) shows that (3) implies the reverse Hölder inequality (2) for all $p < 1 + \frac{\lambda - \varepsilon}{\lambda \rho^{-1}} \frac{1}{\varepsilon}$.

**Proof of Theorem 2.** Set $E = \{x \in Q_0 : f(x) > f^{*}_\mu(t)\}$, and apply the Covering Lemma to $E$ and number $\rho$. We get cubes $Q_i \subset Q_0$, satisfying (i)-(iii). Since $\rho < 1 - \lambda/2$, we obtain from (i) that for each $Q_i$,

$$(f \chi_{Q_i})^{*}_\mu((1 - \lambda/2)\mu(Q_i)) \leq f^{*}_\mu(t).$$

Hence, by Theorem 1,

$$f_{Q_i, \mu} \leq \frac{\lambda}{\lambda - \varepsilon} (f \chi_{Q_i})^{*}_\mu((1 - \lambda/2)\mu(Q_i)) \leq \frac{\lambda}{\lambda - \varepsilon} f^{*}_\mu(t),$$

and so,

$$\Omega_\mu(f; Q_i) \leq \frac{\varepsilon \lambda}{\lambda - \varepsilon} f^{*}_\mu(t).$$

Further, by (ii),

$$\sum_i \mu(Q_i \cap E) \leq B(n) \mu(E) \leq B(n)t.$$
Therefore, using a well-known property of rearrangement (see, e.g. [1]) and (4), (5), we obtain
\[
t(f^{**}_\mu(t) - f^*_\mu(t)) = \int_E (f(x) - f^{**}_\mu(t)) d\mu(x) = \sum_i \int_{E \cap Q_i} (f(x) - f^{**}_\mu(t)) d\mu(x)
\]
\[
= \sum_i \int_{E \cap Q_i} (f(x) - f_{Q_i,\mu}) d\mu(x) + \sum_i \mu(E \cap Q_i) (f_{Q_i,\mu} - f^{**}_\mu(t))
\]
\[
\leq \frac{\varepsilon \lambda}{\lambda - \varepsilon} f^*_\mu(t) \sum_i \mu(Q_i) + \frac{\varepsilon}{\lambda - \varepsilon} f^{**}_\mu(t) \sum_i \mu(E \cap Q_i)
\]
\[
\leq B(n) \frac{\lambda}{\rho} + \frac{1}{\lambda - \varepsilon} \varepsilon f^*_\mu(t),
\]
which gives the desired result. \square

REFERENCES


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