PHY 171
Homework 2 solutions
(Due by beginning of class on Thursday, January 12, 2012)

1. The displacement of a body in simple harmonic motion is given as a function of time by

$$x = (6.00 \text{ m}) \cos\left[(3\pi \text{ rad/s})\, t + \pi/3 \text{ rad}\right]$$

(a) Calculate the displacement at $t = 1.45$ s.

Solution: $x = (6.00 \text{ m}) \cos\left[(3\pi \text{ rad/s})\,(1.45 \text{ s}) + \pi/3 \text{ rad}\right] = \mathbf{-3.26 \text{ m}}$

(b) Calculate the velocity at $t = 1.45$ s.

Solution: $v = \dfrac{dx}{dt} = -3\pi\,(6.00 \text{ m}) \sin\left[(3\pi \text{ rad/s})\,(1.45 \text{ s}) + \pi/3 \text{ rad}\right] = \mathbf{-47.4 \text{ m/s}}$

(c) Calculate the acceleration at $t = 1.45$ s.

Solution: $a = \dfrac{d^2x}{dt^2} = -(3\pi)^2\,(6.00 \text{ m}) \cos\left[(3\pi \text{ rad/s})\,(1.45 \text{ s}) + \pi/3 \text{ rad}\right] = \mathbf{+290 \text{ m/s}^2}$

(d) Calculate the phase at $t = 1.45$ s.

Solution: Phase at $t = 1.45$ s is equal to $\left[3\pi \text{ rad/s}\,(1.45 \text{ s}) + \pi/3 \text{ rad}\right] = \mathbf{14.71 \text{ rad}}$

(e) Based on some/all of your answers above, discuss in a few sentences whether the body is located to the right or left of the zero displacement position ($x = 0$) at $t = 1.45$ s, whether it is moving toward the right (i.e., its velocity vector is directed toward the positive end of the x-axis) or left (i.e., its velocity vector is directed toward the negative end of the x-axis) at $t = 1.45$ s, and whether it is speeding up or slowing down at $t = 1.45s$. As always, to receive full credit, you must provide satisfactory explanations for your answer choices.

Solution:

From part (a), we see that x is negative at $t = 1.45$ s, meaning that the body is located to the left of the zero displacement position at this time.

From part (b), we see that v is also negative at $t = 1.45$ s, meaning that the body is moving toward the negative end of the x-axis.

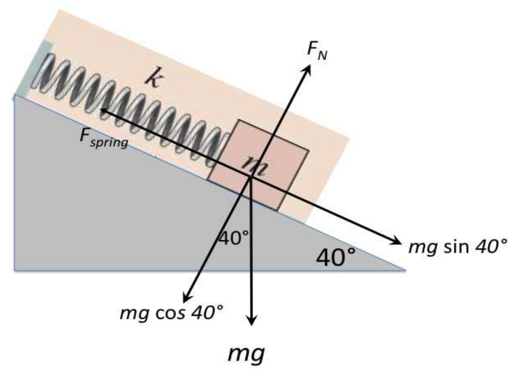
If the body is located to the left of the equilibrium (zero displacement) position moving toward the negative end, we know that it should be slowing down (until it comes to a stop at $x = -A$, where A is the amplitude). This is supported by our answer in part (c), where we find a positive acceleration. We know from PHY 170 that if acceleration and velocity vectors are directed opposite to each other, the object slows down. Moreover, this is SHM, where we know that the acceleration vector should always be directed toward the equilibrium position. If the body is to the left of the equilibrium position, its acceleration vector must be directed toward the right (positive), which is what we find in part (c).

2. A block of mass 1.40 kg is connected to a massless spring of spring constant 120.50 N/m. The other end of the spring is connected to the top of a 40.0° incline.

(a) Assuming that the block slides without friction on the inclined plane, by how much will the spring extend until equilibrium is reached?

Solution: Consider the free body diagram shown below.

The weight mg of the block acts downward as shown. The cosine component of mg is balanced by the normal force of the inclined plane on the block, but it is of no use in the present problem. When the system is in equilbrium, however, the sine component of mg must balance the force of the spring on the block, and we can use this to solve our problem.



If we now consider the x-axis parallel to the incline and positive to the right, then the net force on the block:

$$\sum F = mg \sin 40° - kx_0 = 0$$

where x_0 is the amount by which the spring extends until equilibrium is reached. This gives

$$x_0 = \frac{mg \sin 40°}{k} = \frac{(1.40\,\text{kg})(9.8\,\text{m/s}^2)\sin 40°}{120.50\,\text{N/m}} = \mathbf{0.0732\ m \equiv 7.32\ cm}$$

(b) If the block is now pulled slightly down the incline and released, what is the frequency (in Hz) of the resulting oscillations?

Solution:

If the block is now pulled slightly down the incline and released, it will be just like a block of mass m on a horizontal block in SHM, and so the frequency (in Hz) will be

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{120.50\,\text{N/m}}{1.40\,\text{kg}}}$$

$$\Rightarrow \quad f = \mathbf{1.48\ Hz}$$

3. A block-spring system on a frictionless horizontal surface has a mechanical energy of 1.00 J, an amplitude of 10.0 cm, and a maximum speed of 1.20 m/s.

   Solution: We are given that

$$\text{Total Energy, } E = 1.00 \text{ J}$$

$$\text{Amplitude, } A = 10.0 \times 10^{-2} \text{ m}$$

$$\text{maximum speed, } v_{\text{max}} = 1.20 \text{ m/s}$$

(a) Find the spring constant.

Solution: Since $E = \frac{1}{2}\,kA^2$, we get by cross multiplying that

$$k = \frac{2E}{A^2} = \frac{2\ (1.00 \text{ J})}{(10 \times 10^{-2} \text{ m})^2} = \mathbf{200\ N/m \equiv 2.00 \times 10^2\ N/m}$$

(b) Find the mass of the block.

Solution: Since the total energy remains constant, we have $E = \frac{1}{2}\,mv_{\text{max}}^2$, so we get by cross multiplying that the mass of the block is given by

$$m = \frac{2E}{v_{\text{max}}^2} = \frac{2\ (1.00 \text{ J})}{(1.20 \text{ m/s})^2} = \mathbf{1.39\ kg}$$

(c) Find the frequency of oscillation.

Solution: Since we know that $v_{\text{max}} = \omega\, A$, we can find the frequency using

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\,\frac{v_{\text{max}}}{A} = \frac{1}{2\pi}\left(\frac{1.20 \text{ m/s}}{10.0 \times 10^{-2} \text{ m}}\right) = \mathbf{1.91\ Hz}$$

(d) Find the potential energy when the displacement of the block is 5.0 cm?

Solution: At $x = 5.0$ cm $\equiv 5.0 \times 10^{-2}$ m, the potential energy is given by

$$U = \frac{1}{2}\,kx^2 = \frac{1}{2}\Big(200 \text{ N/m}\Big)\Big(5.0 \times 10^{-2} \text{ m}\Big)^2 = \mathbf{0.25\ J}$$

(e) Find the kinetic energy when the displacement of the block is 5.0 cm? As always, remember to show steps and/or provide explanations for all your answers if you want full credit.

Solution: Since the total mechanical energy is constant, and $E = K + U$, the kinetic energy can be found easily by doing

$$K = E - U = 1.00 \text{ J} - 0.25 \text{ J} = \mathbf{0.75\ J}$$

Note that this is in agreement with our in-class assignment from the discussion session of Jan 10, where we found that at half the amplitude, 1/4 of the energy is potential, and 3/4 is kinetic.

4. An object of mass 0.200 kg is attached to a spring whose spring constant is 80.0 N/m. The object is subject to a resistive force given by $-bv$, where v is its velocity in m/s.

(a) If the damped frequency is 0.995 times the undamped frequency, what is the value of the constant b?

Solution: I'll continue using the same notation in my lecture notes, where I've written the undamped (i.e., SHM) frequency as ω (your text calls this ω_0), and the frequency in the presence of damping as ω'. In this problem, we are given that $\omega' = 0.995\,\omega$. Recall from our lecture notes (equation 14.3.e) that for damped motion, we have

$$(\omega')^2 = \omega^2 - \alpha^2$$

where $\omega^2 = k/m$; here, k is the spring constant. This gives

$$\alpha^2 = \omega^2 - (\omega')^2 = \omega^2 - (0.995\,\omega)^2 = \left[1 - (0.995)^2\right]\omega^2 = \left[1 - (0.995)^2\right]\frac{80.0\text{ N/m}}{0.200\text{ kg}} = 3.99\,(\text{s}^{-1})^2$$

and since $\alpha = b/2m$, we get

$$b = 2m\alpha = 2(0.200\text{ kg})\sqrt{3.99} = \mathbf{0.8\ kg/s}$$

(b) What is the Q of the system?

Solution: $Q = \dfrac{m\omega}{b} = \dfrac{0.200\text{ kg}}{0.800\text{ kg/s}}\sqrt{\dfrac{0.200\text{ N/m}}{0.200\text{ kg}}} = \mathbf{5.0}$

(c) How much time does it take for the amplitude to drop to a fraction $1/e$ of the original amplitude?

Solution: The time taken for amplitude to drop by factor $1/e$ is given by

$$t_L = \frac{1}{\alpha} = \frac{1}{\sqrt{3.99}} = \mathbf{0.5\ s}$$

(d) By what factor is the amplitude of the oscillation reduced after 4 complete cycles?

Solution: After time t s, the amplitude is $A(t) = A\,e^{-\alpha t}$, so if we put t equal to the time for 4 complete cycles, we will obtain our answer. Now, the time for 4 complete cycles is

$$t_4 = 4T = 4\left(\frac{2\pi}{\omega'}\right) = 4\left(\frac{2\pi}{0.995\,\omega}\right) = 4\left(\frac{2\pi}{0.995\,\sqrt{80.0/0.200}}\right) = 1.3\,\text{s}$$

Therefore, the amplitude after 4 cycles is $A_4 = Ae^{-(\sqrt{3.99})\,1.3} = 0.08\,A$, meaning that the amplitude is reduced by a factor of **0.08** or **8/100** after 4 complete cycles.

(e) What fraction of the original energy is left after 4 oscillations?

Solution: Energy goes as square of the amplitude ($E = \frac{1}{2}\,kA^2$), so if amplitude decreases by a factor of 0.08, then the energy must decrease by a factor of $(0.08)^2 = \mathbf{6.4 \times 10^{-3}}$.

5. An object of mass 0.200 kg is attached to a spring whose spring constant is 40.0 N/m. The object is subject to a resistive force given by $-bv$, where v is its velocity (in m/s), and $b = 4.00$ kg/s. The object is subjected to an external force $F(t) = F_0 \sin(\omega_f t)$, where $F_0 = 2.00$ N and $\omega_f = 30.0$ s$^{?1}$. What is the amplitude of the forced oscillation after steady state has been achieved?

   Solution: Given

$$\text{mass, } m = 0.200 \text{ kg}$$

$$\text{spring constant, } k = 40.0 \text{ N/m}$$

$$\text{damping constant, } b = 4.00 \text{ kg/s}$$

$$\text{amplitude of external force, } F_0 = 2.00 \text{ N}$$

$$\text{angular frequency of external force, } \omega_f = 30.0\,\text{s}^{-1}$$

   we get the "natural" angular frequency (of undamped SHM) as

$$\omega^2 = \frac{k}{m} = \frac{40.0 \text{ N/m}}{0.200 \text{ kg}} = 200 \text{ (rad/s)}^2$$

   whereas α is

$$\alpha = \frac{b}{2m} = \frac{4.00 \text{ kg/s}}{2(0.200 \text{ kg})} = 10 \text{ s}^{-1}$$

   The amplitude of forced oscillations can then be calculated by using equation (14.3.p) from the lecture notes:

$$A_f = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_f^2)^2 + (2\alpha\omega_f)^2}}$$

$$= \frac{2/0.2}{\sqrt{(200 - 900)^2 + (2(10)30)^2}}$$

$$\Rightarrow \quad A_f = \mathbf{1.08 \text{ cm}}$$

Submit neat work, with answers or solutions clearly marked by the question number. Unstapled, untidy work will be charged a handling fee of 20% penalty. Writing only an answer without showing the steps used to get to that answer will fetch very few points, even if the answer is correct. Late homework will not be accepted.