Product Rule and Chain Rule Estimates for Fractional Derivatives on Spaces that satisfy the Doubling Condition

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Abstract

The purpose of this paper is to prove some classical estimates for fractional derivatives of functions defined on a Coifman-Weiss space of homogeneous type. In particular the Product Rule and Chain Rule estimates in [KP] and [CW]. The fractional calculus of M. Riesz was extended to these spaces in [GSV]. Our main tools are fractional difference quotients and the square fractional derivative of R. Strichartz in [S] extended to this context. For the particular case of $\mathbb{R}^n$, our approach unifies the proofs of these estimates and clarifies the role of Calderón's formula for these results. Since the square fractional derivative can be easily discretized, we also show that the discrete and continuous Triebel-Lizorkin norms for fractional Sobolev spaces on spaces of homogeneous type are equivalent.

1 Introduction and Definitions

Let (X, ρ, μ) be a space of homogeneous type, which means that ρ is a quasidistance and μ is a doubling measure, such that $\mu(\{x\}) = 0$ and $\mu(\{X\}) = \infty$. Let $\delta_\mu(x,y) = \inf_B \{\mu(B); y, x \in B,\ B\ ball\}$ be the measure quasi-distance. Macias and Segovia have shown in [MS] that there is a quasidistance δ equivalent to δ_μ and a constant A such that

$$|\delta(x,y_1) - \delta(x,y_2)| \leq A\ \delta^\gamma(y_1,y_2)\,\{\delta(x,y_1) + \delta(x,y_2)\}^{1-\gamma}$$

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for some $\gamma, 0 < \gamma \leq 1$ and all x, y_1, y_2. The exponent γ is called the order of the space. A consequence of this result is that the Lipschitz classes $Lip(\alpha) = \left\{ f : \sup\limits_{x \neq y} \frac{|f(x)-f(y)|}{\delta^\alpha(x,y)} < \infty \right\}$ are not trivial for $0 < \alpha \leq \gamma$.

R. R. Coifman has constructed approximations to the identity $s(x, y, t)$ on (X, δ, μ) that satisfy the following properties:

i) $s(x, y, t) = s(y, x, t)$ for all x, y in X and $t > 0$.

ii) $|s(x, y, t)| \leq \frac{c_1}{t}$ for all x, y in X and $t > 0$, $s(x, y, t) = 0$ if $\delta(x, y) > b_1 t$, and $\frac{c_2}{t} < s(x, y, t)$ if $\delta(x, y) < b_2 t$.

iii) $|s(x, y, t) - s(x', y, t)| < c_3 \frac{\delta^\gamma(x,x')}{t^{1+\gamma}}$ for all x, x', y in X and $t > 0$.

iv) $\int s(x, y, t) d\mu(y) = 1$ for all x in X and $t > 0$.

v) $s(x, y, t)$ is continuously differentiable with respect to t.

We will also consider $q(x, y, t) = t \frac{\partial}{\partial t} s(x, y, t)$, which satisfy the following properties

i′) $q(x, y, t) = q(y, x, t)$ for all x, y in X and $t > 0$.

ii′) $q(x, y, t) = 0$ if $\delta(x, y,) > b_1 t$

iii′) $|q(x, y, t)| \leq \frac{c_4}{t}$ for all x, y in X and $t > 0$

iv′) $|q(x, y, t) - q(x', y, t)| \leq c_5 \frac{\delta^\gamma(x,x')}{t^{1+\gamma}}$ for all x, x', y in $X, t > 0$

v′) $\int q(x, y, t) d\mu(y) = 0$ for all x in X and $t > 0$.

We will denote by Q_t the operator

$$Q_t f(x) = -\int_X q(x, y, t) f(y) d\mu(y).$$

In [GSV] it is shown that for each $\alpha > 0$ the function

$$\delta_{-\alpha}(x, y) = \left(\int_0^\infty t^{-\alpha} s(x, y, t) \frac{dt}{t} \right)^{\frac{1}{-\alpha-1}} \quad \text{for } x \neq y$$

and $\delta_{-\alpha}(x,y)=0$ for $x=y$, is a quasidistance equivalent to δ and of the same order.

The fractional derivative of order $\alpha, 0<\alpha<\gamma$, of a function in $Lip(\beta)$, $\alpha<\beta<\gamma$ and bounded support was defined in [GSV] in terms of the hypersingular integral

$$D^\alpha f(x)=\int_X \frac{f(y)-f(x)}{\delta_{-\alpha}^{1+\alpha}(x,y)}d\mu(y).$$

This definition was extended to $L^p, 1<p<\infty$ in [GV1] in the following way: Let $f\in L^1_{loc}$ such that $\int_X \frac{f(x)}{[1+\delta(x_0,x)]^{1+\alpha}}d\mu(x)<\infty$. We say that $D^\alpha f$ exists in L^p if there is a g in L^p such that $<g,\varphi>=<f,D^\alpha\varphi>$ for all $\varphi\in Lip(\beta)$, $\alpha<\beta<\gamma$ and bounded support.

In that paper it is shown that for $f\in \dot{L}^p_\alpha, ||D^\alpha f||_p$ is equivalent to the Triebel-Lizorkin norm $||f||_{\dot{F}^{\alpha,2}_p}=\left\|\left(\int_0^\infty t^{-2\alpha}|Q_tf|^2\frac{dt}{t}\right)^{1/2}\right\|_p$.

Triebel-Lizorkin spaces on spaces of homogeneous type were introduced by Han and Saywer in [HS]. It can also be seen that for $f\in \dot{F}^{\alpha,2}_p, 1<p<\infty$ and α small, $D^\alpha f=\lim_{\substack{a\to 0\\ b\to\infty}} -\frac{1}{\alpha}\int_a^b t^{-\alpha}Q_tf\frac{dt}{t}$ in L^p, see [GV2], but this result won't be needed in this paper.

We will use in this paper another expression for $||D^\alpha f||_p$ based on fractional difference quotients. This expression will be the square fractional derivative of R. Strichartz in $\mathbb{R}^n$, see [S], extended to the spaces of homogeneous type. See also [H] and [M] for the one dimensional case.

Definition 1 *Let $f\in L^1_{loc}(X)$, $0<\alpha<\gamma$, $B_t(x)=\{y:\delta_\mu(x,y)<t\}$ where $\delta_\mu(x,y)$ is the measure quasidistance, we define*

$$S_\alpha f(x)=\left(\int_0^\infty\left[\frac{1}{\mu(B_t(x))^{1+\alpha}}\int_{B_t(x)}|f(y)-f(x)|\,d\mu(y)\right]^2\frac{dt}{t}\right)^{1/2}.$$

$S_\alpha(f)$ will be called the square fractional derivative of order α of f. Note that if $\delta'(x,y)$ is equivalent to $\delta_\mu(x,y)$, the square fractional derivative defined in terms of $\delta'-$ balls is equivalent to S_α. We will denote by M the Hardy-Littlewood maximal function

$$M(h)(x)=\sup_t\frac{1}{\mu(B_t(x)}\int_{B_t(x)}|h(y)|d\mu(y).$$

The letters $c, C, C_1, C_2, ...,$ will denote positive constants not always the same, $\bar{\alpha} < \gamma$ will denote a fixed constant that depends on the space.

2 Statement of the results

Theorem 1 *Let* $f(x) \in \dot{F}_p^{\alpha,2}, 1 < p < \infty,\ 0 < \alpha < \bar{\alpha},\ h \in L^q,\ 1 < q \leq \infty, 1 < r < \infty,$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, *then there is a constant* C *independent of* f *and* h *such that*

$$\left|\left|\left(\int_0^\infty \left[\frac{1}{\mu(B_t(x))^{1+\alpha}} \int_{B_t(x)} |f(y) - f(x)||h(y)|d\mu(y)\right]^2 \frac{dt}{t}\right)^{1/2}\right|\right|_r \leq C||h||_q||f||_{\dot{F}_p^{\alpha,2}}. \tag{1}$$

Theorem 2 *Let* $1 < p < \infty,\ 0 < \alpha < \bar{\alpha}$, *then there are constants* C_1 *and* C_2 *such that*

$$C_1||f||_{\dot{F}_p^{\alpha,2}} \leq ||S_\alpha(f)||_p \leq C_2||f||_{\dot{F}_p^{\alpha,2}}. \tag{2}$$

Theorem 3 *Product Rule. Let* $0 < \alpha < \bar{\alpha},\ 1 < p_1, p_2 < \infty,\ 1 < q_1,\ q_2 \leq \infty,\ 1 < r < \infty,$ *and* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, *then there is a constant* C *independent of* f *and* g *such that*

$$||D^\alpha(fg)||_r \leq C||D^\alpha f||_{p_1}||g||_{q_1} + C||D^\alpha g||_{p_2}||f||_{q_2}. \tag{3}$$

Theorem 4 *Let* $1 < p < \infty,\ 0 < \alpha < \bar{\alpha}$, *then* $L_\alpha^p \cap L^\infty$ *is closed under pointwise multiplication.*

Theorem 5 *Chain Rule.*

a) Let $F \in C^1(\mathbb{C}),\ ||F'||_\infty \leq K$ *and* $g : X \to \mathbb{C},\ 0 < \alpha < \bar{\alpha}, 1 < p < \infty,$ *then there is a constant* C *independent of* F *and* g *such that*

$$||D^\alpha(F(g))||_p \leq CK||D^\alpha g||_p \tag{4}$$

b) Let $F \in C^1(\mathbb{C})$, *and* $H(z) \geq 0$, *such that* $\int_0^1 |F'(sz_1 + (1-s)z_2|ds \leq c\,[H(z_1) + H(z_2)]$ *for all* z_1, z_2. *If* $1 < r, p, q < \infty$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, *and* $0 < \alpha < \bar{\alpha}$, *then there is a constant* C *independent of* F, H *and* g *such that*

$$||D^\alpha(F(g)||_r \leq C||H(g)||_q||D^\alpha g||_p. \tag{5}$$

Corollary of Theorem 5. *Power Rule*

Let f in $\dot{F}_p^{\alpha,2}$ and $f^{k-1} \in L^p$, $k \geq 2, 0 < \alpha < \bar{\alpha}, 1 < p < \infty, 1 < q \leq \infty$, $1 < r < \infty$, and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ then

$$\left|\left|D^\alpha f^k\right|\right|_r \leq C||kf^{k-1}||_q||D^\alpha f||_p. \tag{6}$$

Theorem 6 *Let* $1 < p < \infty$, $0 < \alpha < \bar{\alpha}$ *and*

$$||f||_{\dot{F}_{p,d}^{\alpha,2}} = ||\left(\sum_{k=-\infty}^{\infty} 2^{2k\alpha}|Q_{2^{-k}}f|^2\right)^{1/2}||_p \tag{7}$$

then, $f \in \dot{F}_{p,d}^{\alpha,2}$ *if and only if* $f \in \dot{F}_p^{\alpha,2}$ *and the corresponding norms are equivalent.*

3 Proofs

Proof of Theorem 1

Estimates like (1), though not exactly in this form, are known in $\mathbb{R}^n$. It can be thought of as Calderon's commutator estimate for the square fractional derivative of order α. (see [C] Theorem 2.) The argument that is needed to prove it is essentially contained in the proof of Christ and Weinstein of the Chain Rule in $\mathbb{R}$ in [CW], see also [T]. We will redo their argument in its continuous form and with the modifications needed in our case, in particular the use of Calderón's formula proved by Han and Sawyer in [HS]. Let $\tilde{Q}_t^N = \int_{1/N}^N Q_{ts}\frac{ds}{s}$, Han and Sawyer have shown in [H-S] (their proof is also valid for the continuous version) that for N sufficiently large and $0 < \alpha < \bar{\alpha}$, the operator

$$T_N = \int_0^\infty \tilde{Q}_t^N Q_t \frac{dt}{t}$$

is invertible in $\dot{F}_p^{\alpha,2}$, therefore for N large and fixed, $\tilde{Q}_t = \tilde{Q}_t^N$, and f in $\dot{F}_p^{\alpha,2}$ we have

$$f = \int_0^\infty \tilde{Q}_t Q_t (T_N^{-1} f)\frac{dt}{t} \tag{8}$$

in $\dot{F}_p^{\alpha,2}$, and for $f \in M^{\beta,\varepsilon}$ the class of test functions introduced in [H-S], the representation is pointwise. Since $M^{\beta,\varepsilon}$ is dense in $\dot{F}_p^{\alpha,2}$ we shall assume without loss of generality that $f \in M^{\beta,\varepsilon}$, the general case follows by approximation. Note that $\tilde{Q}_t$ also satisfies i′), ii′), iii′), iv′), and v′) with constants that depend only on N.

We will first obtain an estimate for

$$\frac{1}{\mu(B_t(x))}\int_{B_t(x)} |f(y)-f(x)||h(y)|d\mu(y). \tag{9}$$

where $B_t(x)=\{y:\delta(x,y)<t\}$. Let χ be the characteristic function of the interval $[0,1]$, and rewriting f using (8) we can majorize (9) by

$$\frac{1}{\mu(B_t(x))}\int_X\int_0^\infty |\tilde{Q}_sQ_s(T_N^{-1}f)(y)-\tilde{Q}_sQ_s(T_N^{-1}f)(x)|\frac{ds}{s}|h(y)|\chi\left(\frac{\delta(x,y)}{t}\right)d\mu(y)$$

changing the order of integration, the last expression is equal to

$$\int_0^\infty \frac{1}{\mu(B_t(x))}\int_X |\tilde{Q}_sQ_s(T_N^{-1}f)(y)-\tilde{Q}_sQ_s(T_N^{-1}f)(x)||h(y)|\chi\left(\frac{\delta(x,y)}{t}\right)d\mu(y)\frac{ds}{s}$$

$$=\int_0^t [\ ]\frac{ds}{s}+\int_t^\infty [\ ]\frac{ds}{s}=I_1+I_2$$

To estimate I_1 we use properties ii′) and iii′) of $\tilde{Q}_s$ to get

$$I_1\le\int_0^t\frac{1}{\mu(B_t(x))}\int_{B_t(x)}\left\{M(Q_s(T_N^{-1}f))(y)+M(Q_s(T_N^{-1}f))(x)\right\}|h(y)|d\mu(y)\frac{ds}{s}$$

$$\le\int_0^t 2\,M\left[M(Q_s(T_N^{-1}f))\cdot M(h)\right](x)\frac{ds}{s}.$$

To estimate I_2, observe first that using property ii′), vi′) of $\tilde{Q}_s$ and the fact that $\delta(x,y)<t<s$, we have

$$\left|\tilde{Q}_s(Q_s(T_N^{-1}f))(y)-\tilde{Q}_s(Q_s(T_N^{-1}f)(x))\right|\le c\frac{\delta^\gamma(x,y)}{s^\gamma}M(Q_s(T_N^{-1}f))(x)$$

therefore

$$\begin{aligned}
I_2 &\le \int_t^\infty\frac{1}{\mu(B_t(x))}\int_{B_t(x)}c\frac{\delta^\gamma(x,y)}{s^\gamma}(MQ_s(T_N^{-1}f))(x)|h(y)|d\mu(y)\frac{ds}{s}\\
&\le \int_t^\infty\left(\frac{t}{s}\right)^\gamma M(Q_s(T_N^{-1}f))(x)\cdot M(h)(x)\frac{ds}{s}\\
&\le \int_t^\infty\left(\frac{t}{s}\right)^\gamma M\left[M(Q_s(T_N^{-1}f))\cdot M(h)\right](x)\frac{ds}{s}
\end{aligned}$$

Let's write now $g(s) = M\left[M(Q_s(T_N^{-1}f)\cdot M(h)\right](x)$, observe that from the estimates above we have

$$(9) \leq \int_0^t g(s)\frac{ds}{s} + \int_t^\infty \left(\frac{t}{s}\right)^\gamma g(s)\frac{ds}{s}. \tag{10}$$

We now want to estimate

$$J = \left(\int_0^\infty t^{-2\alpha}\left[\int_0^t g(s)\frac{ds}{s} + \int_t^\infty \left(\frac{t}{s}\right)^\gamma g(s)\frac{ds}{s}\right]^2 \frac{dt}{t}\right)^{1/2}$$

We perform the change of variables $s = tv$ in the integrals with respect to s, and let $\zeta(v) = 1$ for $0 < v \leq 1$, $\zeta(v) = v^{-\gamma}$ for $1 < v$, we then have

$$J \leq \left(\int_0^\infty \left[\int_0^\infty t^{-\alpha}\zeta(v)g(tv)\frac{dv}{v}\right]^2 \frac{dt}{t}\right)^{1/2}.$$

We next apply Minkowsky's inequality to majorize the last expression by

$$\int_0^\infty \zeta(v)\left(\int_0^\infty t^{-2\alpha}g^2(tv)\frac{dt}{t}\right)^{1/2}\frac{dv}{v} =$$

$$\int_0^\infty \zeta(v)v^\alpha\left(\int_0^\infty (tv)^{-2\alpha}g^2(tv)\frac{dt}{t}\right)^{1/2}\frac{dv}{v} \leq$$

$$C\left(\int_0^\infty l^{-2\alpha}g^2(l)\frac{dl}{l}\right)^{1/2}$$

because $0 < \alpha < \gamma$.

From (10), the estimate above for J and the fact that $ct \leq \mu(B_t(x))$ we then have

$$\left(\int_0^\infty \left[\frac{1}{\mu(B_t(x)^{1+\alpha}}\int_{B_t(x)} |f(y)-f(x)||h(y)|d\mu(y)\right]^2 \frac{dt}{t}\right)^{1/2} \leq$$

$$C\left(\int_0^\infty l^{-2\alpha}\left[M\left[M(Q_l(T_N^{-1}f)\cdot M(h)\right](x)\right]^2\frac{dl}{l}\right)^{1/2}.$$

To conclude the proof of the theorem we have to estimate the $L^r - norm$ of the last expression. We use the Fefferman-Stein theorem to obtain first that the $L^r - norm$ of this expression is less than or equal to

$$\left\| \left(\int_0^\infty l^{-2\alpha} |M(Q_l(T_N^{-1}f) \cdot M(h)|^2 \frac{dl}{l} \right)^{1/2} \right\|_r ,$$

and since $M(h)$ is independent of l we apply Holder's inequality with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ to majorize the last expression by

$$||M(h)||_q \, || \left(\int_0^\infty l^{-2\alpha} [M(Q_l(T_N^{-1}f))]^2 \frac{dl}{l} \right)^{1/2} ||_p .$$

We use the Fefferman-Stein theorem again and the Hardy-Littlewood theorem to obtain that the former expression is less than or equal to

$$C||h||_q ||T_N^{-1}f||_{\dot{F}_p^{\alpha,2}} .$$

Finally, the fact that T_N^{-1} is bounded in $\dot{F}_p^{\alpha,2}$ concludes the proof of the theorem.

<u>Proof of Theorem 2</u>

In the particular case of $\mathbb{R}^n$, this result is due to Strichartz in [S].

We prove first the left hand side inequality. Let $f \in L^1_{loc}$, using the properties of $q(x,y,t)$, and the fact that $\mu(B_t(x)) \leq ct$, we have

$$\left| \int_X f(y) q(x,y,t) d\mu(y) \right| = \left| \int_X [f(y) - f(x)] q(x,y,t) d\mu(y) \right|$$

$$\leq \frac{c_4}{t} \int_{B_{b_1 t}(x)} |f(y) - f(x)| \, d\mu(y) \leq C \frac{1}{\mu(B_{b_1 t}(x))} \int_{B_{b_1 t}(x)} |f(y) - f(x)| \, d\mu(y)$$

Since $\mu(B_t(x)) \leq ct$ we then have

$$\left(\int_0^\infty t^{-2\alpha} |Q_t f|^2 \frac{dt}{t} \right)^{1/2} \leq C_1 \left(\int_0^\infty \left[\frac{1}{\mu(B_t(x))^{1+\alpha}} \int_{B_t(x)} |f(y) - f(x)| dy \right]^2 \frac{dt}{t} \right)^{1/2}$$

that implies the desired inequality.

To prove the right hand side, we can assume that $||f||_{\dot{F}_p^{\alpha,2}}$ is finite, and use Theorem 1 with $h(y) \equiv 1$ and $q = \infty$. This concludes the proof of Theorem 2.

Proof of Theorem 3

We will show first that

$$||S_\alpha(fg)||_r \leq C||f||_{\dot{F}_{p_1}^{\alpha,2}}||g||_{q_1} + C||S_\alpha g||_{p_2}||f||_{q_2}. \tag{11}$$

Observe that we have the following inequality for the fractional difference quotients,

$$\frac{1}{\mu(B_t(x))^{1+\alpha}} \int_{B_t(x)} |(fg)(y) - (fg)(x)| d\mu(y) \leq$$

$$\frac{1}{\mu(B_t(x))^{1+\alpha}} \int_{B_t(x)} |f(y)-f(x)||g(y)|d\mu(y)+|f(x)|\frac{1}{\mu(B_t(x)^{1+\alpha}} \int_{B_t} |g(y)-g(x)|d\mu(y).$$

Computing first the $L^2-norms$ with respect to $\frac{dt}{t}$ and then applying Theorem 1 to the first term and Holder's inequality to the second term we get (11). Finally (3) follows from (11) using Theorem 2 and the fact that $||D^\alpha h|| \approx ||h||_{\dot{F}_p^{\alpha,2}}$.

Proof of Theorem 4

Clearly fg is in $L^p \cap L^\infty$. The fact that $D^\alpha(fg)$ is in L^p follows from Theorem 3 with $q_1 = q_2 = \infty$, $p_1 = p_2 = p$.

Proof of Theorem 5

a) We will estimate $S_\alpha(F(g))$. Note that

$$\frac{1}{\mu(B_t(x))^{1+\alpha}} \int_{B_t(x)} |F(g(y)) - F(g(x))| \, d\mu(y) \leq$$

$$\frac{1}{\mu(B_t(x))^{1+\alpha}} \int_{B_t(x)} ||F'||_\infty \; |g(y) - g(x)|d\mu(y) \leq$$

$$\leq \frac{K}{\mu(B_t(x))^{1+\alpha}} \int_{B_t(x)} |g(y) - g(x)|d\mu(y).$$

Therefore

$$S_\alpha(F(g))(x) \leq K \; S_\alpha(g)(x)$$

which implies (4).

b) We will estimate $S_\alpha(F(g))$ in the following way:

$$\frac{1}{\mu(B_t(x))^{1+\alpha}}\int_{B_t(x)}|F(g(y))-F(g(x))|d\mu(y)\leq$$

$$\frac{1}{\mu(B_t(x))^{1+\alpha}}\int_{B_t(x)}\left|\int_0^1 F'(sg(y)+(1-s)g(x))(g(y)-g(x))ds\right|d\mu(y)\leq$$

$$\frac{1}{\mu(B_t(x))^{1+\alpha}}\int_{B_t(x)}c\left[H(g(y))+H(g(x))\right]|g(y)-g(x)|\,d\mu(y)\leq$$

$$\leq\frac{c}{\mu(B_t(x))^{1+\alpha}}\int_{B_t(x)}H(g(y))\,|g(y)-g(x)|\,d\mu(y)+ H(g(x))\frac{c}{\mu(B_t(x))^{1+\alpha}}\int_{B_t(x)}|g(y)-g(x)|d\mu(y).$$

We now compute the $L^2-norms$ with respect to $\frac{dt}{t}$ and apply Theorem 1 to the first term and Hölder's inequality to the second term to obtain (2.5).

Proof of the Corollary of Theorem 5

Observe that if $F(z)=z^k,\ k\geq 2$, and $H(z)=k\,|z|^{k-1}$, then

$$\int_0^1|F'(sz_1+(1-s)z_2)|\,ds\leq\int_0^1\left[\left(\sum_{j=0}^{k-1}k-1 \atop j|sz_1|^{k-1-j}|(1-s)z_2|^j\right)\right]ds$$

$$\leq c\left[H(z_1)+H(z_2)\right].$$

Therefore (6) follows from (5).

Proof of Theorem 6

Let's see first that $S_\alpha(f)$ can be discretized. We define

$$S_{\alpha d}(f)(x)=\left(\sum_{k=-\infty}^{\infty}\left[\frac{1}{\mu(B_{2^{-k}}(x))^{1+\alpha}}\int_{B_{2^{-k}}(x)}|f(y)-f(x)|d\mu(y)\right]^2\right)^{1/2}.$$

Consider now t such that, $2^{-k} < t \leq 2^{-k+1}$, we have

$$\frac{1}{\mu(B_{2^{-k}}(x))^{1+\alpha}} \int_{B_{2^{-k}}(x)} |f(y) - f(x)| d\mu(y) \leq$$

$$\left(\frac{\mu(B_t(x))}{\mu(B_{2^{-k}}(x))} \right)^{1+\alpha} \frac{1}{\mu(B_t(x))^{1+\alpha}} \int_{B_t(x)} |f(y) - f(x)| d\mu(y)$$

$$\leq c \frac{1}{\mu(B_t(x))^{1+\alpha}} \int_{B_t(x)} |f(y) - f(x)| d\mu(y) \leq c \left(\frac{\mu(B_{2^{-k+1}}(x))}{\mu(B_t(x))} \right)^{1+\alpha} \left(\frac{1}{B_{2^{-k+1}}(x)} \right)^{1+\alpha}.$$

$$\int_{B_{2^{-k+1}}} |f(y) - f(x)| d\mu(y) \leq c' \frac{1}{\mu(B_{2^{-k+1}}(x))^{1+\alpha}} \int_{B_{2^{-k+1}}(x)} |f(y) - f(x)| d\mu(y).$$

In addition $\int_{2^{-k}}^{2^{-k+1}} \frac{dt}{t} = \ln 2$, therefore there are C_1 and C_2. such that

$$C_1 S_{\alpha,d}(f)(x) \leq S_\alpha(f)(x) \leq C_2 S_{\alpha,d}(f)(x).$$

The proof of the left hand side inequality of (1) also proves that

$$\left\| \left(\sum_{k=-\infty}^{\infty} 2^{2k\alpha} |Q_{2^{-k}} f(x)|^2 \right)^{1/2} \right\|_p \leq C_3 || \; S_{\alpha,d}(f)(x)||_p.$$

On the other hand the discrete version of the inequality (1) with $h(y) \equiv 1$, see [CW], shows that $||S_{\alpha,d}(f)||_p \leq C_3 ||f||_{\dot{F}^{\alpha,2}_{p,d}}$. This concludes the proof of Theorem 6.

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