On fractional calculus associated to doubling and non-doubling measures

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Dedicated to Stephen Vági

Abstract. In this paper we present several proofs on the extension of M. Riesz fractional integration and differentiation to the contexts of spaces of homogeneous type and measure metric spaces with non-doubling measures.

1. Introduction, some definitions, and a basic lemma

Professor M. Ash asked me to write a survey article on some of the results that Stephen Vági and I obtained in the nineties on fractional calculus on spaces of homogeneous type. Since Professor Korányi has done an excellent job stating these results in his article in this volume, I thought it would be appropriate to add some proofs and extensions of these results to the recently open area of research of measure metric spaces associated to non-doubling measures.

The notion of space of homogeneous type is due to Coifman and Weiss[**CW**] and it consists of the triple X a non-empty set; $\rho : X \times X \to \mathbb{R}_{\geq 0}$ a quasi-distance, that is $\rho(x,y) = \rho(y,x) \geq 0$, $\rho(x,y) = 0$ if and only if $x = y$, and $\rho(x,z) \leq k(\rho(x,y) + \rho(y,z))$ where k is a positive constant; and a measure μ that satisfies the doubling condition, i.e. $\mu(B_{2r}(x)) \leq C_\mu B_r(x)$ where $B_r(x)$ denotes the ball of radius r and center x and C_μ is independent of x and r. Any space of homogeneous type can be normalized by introducing the measure quasidistance

$$\delta_\mu(x,y) = \inf_B \{\mu(B) : x, y \in B\}$$

where B denotes a ρ-ball. The topologies induce by ρ and δ_μ are equivalent, but the measure quasidistance satisfies the Ahlfors-David property: Let now $B_r(x)$ be a δ_μ-ball of radius r and center x; then there are constants c_1 and c_2 such that

$$c_1 r \leq \mu(B_r(x)) \leq c_2 r. \tag{1.1}$$

Furthermore, in any space of homogeneous type there exists a quasidistance δ equivalent to δ_μ that also satisfies: There is $\gamma, 0 < \gamma \leq 1$, and a constant M such

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that for all x, x' and y in X,

$$|\delta(x,y) - \delta(x',y)| \leq M\ \delta^{\gamma}(x,x')\left[\delta(x,y)+\delta(x',y)\right]^{1-\gamma} \tag{1.2}$$

The space (X, δ, μ) with δ satisfying (1.1) and (1.2) will be called a normalized space of homogeneous type of order γ.

Note that a metric satisfies (1.2) with $\gamma = 1$ and constant $M = 1$. There are many well known examples of spaces of homogeneous type.

On the other hand, a non-homogeneous space or non-doubling measure metric space is a metric space (X, d) with a measure μ that satisfies the growth condition

$$\mu\left(B_r(x)\right) \leq Ar^n \tag{1.3}$$

for some real number $n > 0$ and a constant A independent of x and r. A measure that satisfies (1.3) is also called a non-doubling measure of dimension n.

Note that the measure quasidistance δ_μ introduced before on spaces of homogeneous type satisfies (1.3) with $n = 1$.

The particular case of $\mathbb{R}^n$ with a non-doubling measure is by far the most important case. Calderón-Zygmund operators and in particular the Cauchy integral with respect to non-doubling measures have been studied by several authors: Nazarov, Treil, Volberg, Melnikov, Verdera, Tolsa, Garcia-Cuerva, and the author.

Whenever we can give a single proof that covers both cases we will do it, but there are results that are valid on spaces of homogeneous type that are not known yet for non-homogeneous spaces.

Lemma 1. *Let X be a non-empty set, ρ a quasidistance and μ a measure that satisfies the growth condition (1.3). Then*

(1) $\int_{\rho(x,y)<r} \frac{1}{d(x,y)^{n-\alpha}} d\mu(y) \leq B_1 r^{\alpha}$, $0 < \alpha$
(2) $\int_{\rho(x,y)\geq r} \frac{1}{d(x,y)^{n+\alpha}} d\mu(y) \leq B_2 r^{-\alpha}$, $0 < \alpha$
(3) $\int_{r\leq\rho(x,y)<2r} \frac{1}{d(x,y)^{n}} d\mu(y) \leq B_3$

Proof. To prove (1) we write

$$\begin{aligned}
\int_{\rho(x,y)<r} \frac{1}{d(x,y)^{n-\alpha}} d\mu(y) &= \sum_{k=0}^{\infty} \int_{2^{-k-1}r\leq\rho(x,y)<2^{-k}r} \frac{1}{d(x,y)^{n-\alpha}} d\mu(y) \\
&\leq \sum_{k=0}^{\infty} \frac{1}{\left(2^{-k-1}r\right)^{n-\alpha}} \mu\left(B_{2^{-k}r}(x)\right) \\
&\leq Ar^d 2^{n-\alpha} \sum_{k=0}^{\infty} \left(2^{-k}\right)^{\alpha} \leq B_1 r^{\alpha}
\end{aligned}$$

To prove (2), we write

$$\begin{aligned}
\int_{\rho(x,y)\geq r} \frac{1}{d(x,y)^{n+\alpha}} d\mu(y) &= \sum_{k=0}^{\infty} \int_{2^{k}r\leq\rho(x,y)<2^{k+1}r} \frac{1}{d(x,y)^{n+\alpha}} d\mu(y) \\
&\leq \sum_{k=0}^{\infty} \frac{1}{\left(2^{k}r\right)^{n+\alpha}} \mu\left(B_{2^{k+1}r}(x)\right) \\
&\leq \frac{A2^n}{r^{\alpha}} \sum_{k=0}^{\infty} \frac{c_3}{\left(2^{k}\right)^{\alpha}} \leq B_2 r^{-\alpha}
\end{aligned}$$

Finally to prove (3), we write

$$\int_{r\leq\rho(x,y)<2r}\frac{1}{d\left(x,y\right)^{n}}d\mu\left(y\right)\leq\frac{1}{r^{n}}\mu\left(B_{2r}\left(x\right)\right)\leq A2^{n}=B_{3}.$$

□

2. Hardy-Littlewood-Sobolev Theorem

Our first theorem is a general version of the Hardy-Littlewood-Sobolev Theorem. It appeared in [**GV**] for the doubling case, and in [**GG1**] for the non-doubling case.

THEOREM 1. *Let X be a non-empty set, ρ a quasidistance, and μ a measure without atoms. Let $s>0$, and*

$$I^{(s)}f\left(x\right)=\int_{X}\frac{f\left(y\right)}{\rho^{s}\left(x,y\right)}d\mu\left(y\right).$$

There are $p>1$ and $q>p$ such that $\left\|I^{(s)}f\right\|_{q}\leq C\left\|f\right\|_{p}$ if and only if μ satisfies the growth condition (1.3), $s=n-\alpha$ with $0<\alpha<n$, $1<p<\frac{n}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.

PROOF. We will prove first the necessity part. We assume $\left\|I^{(s)}f\right\|_{q}\leq C\left\|f\right\|_{p}$ and we will show (1.3); i.e.: there are $n>0$ and $A>0$ such that $\mu\left(B_{r}\left(x\right)\right)\leq Ar^{n}$. If $\mu\left(B_{r}\left(x\right)\right)=0$ there is nothing to prove. Let $\mu\left(B_{r}\left(x\right)\right)\neq0$, and let $\chi_{B}\left(y\right)$ be the characteristic function of $B_{r}\left(x\right)$. For each $u\in B_{r}\left(x\right)$, we have

$$I^{(s)}\left(\chi_{B}\right)\left(u\right)=\int_{B_{r}(x)}\frac{1}{\rho\left(u,y\right)^{s}}d\mu\left(y\right)\geq\frac{1}{\left(2kr\right)^{s}}\mu\left(B_{r}\left(x\right)\right).$$

Then using the hypothesis we get

$$\frac{1}{\left(2kr\right)^{s}}\mu\left(B_{r}\left(x\right)\right)^{1+\frac{1}{q}}\leq C\mu\left(B_{r}\left(x\right)\right)^{\frac{1}{p}},$$

which is equivalent to

$$\mu\left(B_{r}\left(x\right)\right)\leq Ar^{n},$$

with $n=s/\left(1+\frac{1}{q}-\frac{1}{p}\right)$. Let now $\alpha=n\left(\frac{1}{p}-\frac{1}{q}\right)$, then $0<\alpha<n$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and $1<p<\frac{n}{\alpha}$.

Next we will show the sufficiency part. We will adopt the standard notation

$$I_{\alpha}f\left(x\right)=I^{(n-\alpha)}f\left(x\right)=\int\frac{f\left(y\right)}{\rho^{n-\alpha}\left(x,y\right)}d\mu\left(y\right).$$

It suffices to show that for $1\leq p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ we have

$$\mu\left(\left\{x\in X:\left|I_{\alpha}f\left(x\right)\right|>\lambda\right\}\right)\leq C\left(\frac{\left\|f\right\|_{L^{p}(\mu)}}{\lambda}\right)^{q},\tag{2.1}$$

for then the strong type estimate follows from the Marcinkiewicz interpolation theorem.

To prove (2.1) we will adapt to our context the proof given by E. Stein in [**S2**]. We can take $f\geq0$ and assume that $\left\|f\right\|_{L^{p}(\mu)}=1$. We have

$$\begin{aligned}I_{\alpha}f\left(x\right)&=\int_{B_{r}(x)}\frac{f\left(y\right)}{d\left(x,y\right)^{n-\alpha}}d\mu\left(y\right)+\int_{B_{r}(x)^{c}}\frac{f\left(y\right)}{d\left(x,y\right)^{n-\alpha}}d\mu\left(y\right)\\&=I+II.\end{aligned}$$

To estimate II we apply Holder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$. Then

$$II \leq \left(\int_{B_r(x)^c} \frac{1}{d\left(x,y\right)^{(n-\alpha)p'}} d\mu\left(y\right)\right)^{1/p'} \leq Cr^{\alpha-\frac{n}{p}},$$

by Lemma 1 and

$$\left(-\left(n-\alpha\right)p'+n\right)\frac{1}{p'} = \alpha - n + n\left(1-\frac{1}{p}\right) = \alpha - \frac{n}{p}.$$

Now for each fixed λ, we have

$$\left\{x \in X : I_\alpha f\left(x\right) > \lambda\right\} \subset \left\{x : I > \frac{\lambda}{2}\right\} \cup \left\{x : II > \frac{\lambda}{2}\right\}.$$

Choosing r such that $Cr^{\alpha-\frac{n}{p}} = \frac{\lambda}{2}$, the second set of the right hand side above is empty and consequently, all we have to do is to show that for that r, $I \leq C\lambda^{-q}$. Using Holder's inequality and Lemma 1, we have

$$\begin{aligned} I &= \int_{B_r(x)} \frac{f\left(y\right)}{d\left(x,y\right)^{n-\alpha}} d\mu\left(y\right) \\ &= \int_{B_r(x)} \frac{f\left(y\right)}{d\left(x,y\right)^{(n-\alpha)\frac{1}{p}}} \cdot \frac{1}{d\left(x,y\right)^{(n-\alpha)\frac{1}{p'}}} d\mu\left(y\right) \\ &\leq \left(\int_{B_r(x)} \frac{f^p\left(y\right)}{d\left(x,y\right)^{(n-\alpha)}} d\mu\left(y\right)\right)^{\frac{1}{p}} \left(\int_{B_r(x)} \frac{1}{d\left(x,y\right)^{(n-\alpha)}} d\mu\left(y\right)\right)^{\frac{1}{p'}}. \end{aligned}$$

Thus

$$\frac{I^p}{\left(\frac{\lambda}{2}\right)^p} \leq \frac{r^{\frac{\alpha p}{p'}}}{\left(\frac{\lambda}{2}\right)^p} \int_{B_r(x)} \frac{f^p\left(y\right)}{d\left(x,y\right)^{(n-\alpha)}} d\mu\left(y\right)$$

and

$$\begin{aligned} \mu\left\{x : I > \frac{\lambda}{2}\right\} &= \mu\left\{x : I^p > \left(\frac{\lambda}{2}\right)^p\right\} \\ &\leq \int_X \frac{r^{\frac{\alpha p}{p'}}}{\left(\frac{\lambda}{2}\right)^p} \int_{B_r(x)} \frac{f^p\left(y\right)}{d\left(x,y\right)^{(n-\alpha)}} d\mu\left(y\right) d\mu\left(x\right) \\ &= \frac{r^{\frac{\alpha p}{p'}}}{\left(\frac{\lambda}{2}\right)^p} \int_X f^p\left(y\right) \left(\int_{B_r(y)} \frac{1}{d\left(x,y\right)^{(n-\alpha)}} d\mu\left(x\right)\right) d\mu\left(y\right) \\ &\leq C\lambda^{-q}, \end{aligned}$$

by using Lemma 1 and

$$\frac{r^{\alpha+\alpha\frac{p}{p'}}}{\lambda^p} = \frac{c\lambda^{\frac{-\alpha p}{\alpha-\frac{n}{p}}}}{\lambda^p} = c\lambda^{\frac{1}{\frac{\alpha}{n}-\frac{1}{p}}} = c\lambda^{-q}.$$

This concludes the proof of Theorem 1. □

3. Fractional integrals a fractional derivatives on Lipschitz spaces

We will introduce next fractional differentiation and prove a boundedness result on appropriate function spaces on the support of the measure. We recall the definition of Lipschitz spaces. Let (X, d, μ) be a measure metric space; a function $f(x)$ is said to be a Lipschitz function of order α, $0 < \alpha < 1$, when there is a constant C such that

$$|f(x) - f(y)| \leq cd^{\alpha}(x, y)$$

for all x, y in the support of μ. Of course the support of μ has to be well defined, where supp(μ) is the smallest closed set F such that for all Borel sets $E, E \subset F^c$, $\mu(E) = 0$. For example, if X is separable, then the support of μ is well defined. The Lipschitz norm of f is defined to be the infimum of the constants c above.

For normalized spaces of homogeneous type of order γ , following Macias and Segovia, we define Lipschitz functions as above but using the measure quasidistance quasidistance δ_{μ},(or the equivalent quasidistance δ) and $0<\alpha \leq \gamma$. It was shown in [**MS**] that the Lipschitz classes for these values of α are not trivial spaces, that is, they have functions not identically zero. We will abuse the notation and also write d for the quasidistance δ that satisfies (1.1) and (1.2), to avoid rewriting formulas when this is the only change.

Let now (X, d, μ) be a non-homogeneous space or a normalized space of homogeneous type of order γ, $0 < \gamma \leq 1$. Note that $\gamma = 1$ when d is a metric and that $n = 1$ when d is a quasidistance. Letting f be a bounded Lipschitz function of order β, $0 < \beta \leq \gamma \leq 1$, we define the fractional derivative of order α, $0 < \alpha < \beta$, of f as

$$D^{\alpha} f(x) = \int_X \frac{f(y) - f(x)}{d^{\alpha+n}(x, y)} d\mu(y).$$

It is not hard to see using Lemma 1 that the integral converges absolutely for all x. This definition extends a well known formula for the fractional powers of the Laplacian $(0 < \alpha < 2)$ on $\mathbb{R}^n$ with Lebesgue measure. The definition can be modified in a standard way to be valid for any Lipschitz function of order β, i.e.

$$\tilde{D}^{\alpha} f(x) = \int_X \frac{f(y) - f(x)}{d^{\alpha+n}(x, y)} - \frac{f(y) - f(x_0)}{d^{\alpha+n}(x_0, y)} d\mu(y) \tag{3.1}$$

where x_0 is any fixed point in X.

LEMMA 2. *Let $s < 0$ and $h > 1$, and d be a metric. Then there exists a positive constant $C_{h,s}$ such that*

$$|d^s(x, z) - d^s(y, z)| \leq C_{h,s} d(x, y) d^{s-1}(x, z)$$

for $hd(x, y) < d(x, z)$.

Note: A similar lemma is true for normalized spaces of homogeneous type of order γ, *i.e.* :There are constants $k_s$and $C_{k,s}$ such that

$$|d^s(x, z) - d^s(y, z)| \leq C_{h,s} d^{\gamma}(x, y) d^{s-\gamma}(x, z)$$

$k_s d(x, y) < d(x, z)$.

We leave the proof of this lemma to the reader. See also [**GV**].

We will prove now the following boundedness result for fractional derivatives.

Theorem 2. *Let (X,d,μ) be a non-homogeneous space or a normalized space of homogeneous type of order γ, $0<\gamma\leq 1$. Let f be a Lipschitz function of order β, $0<\alpha<\beta\leq\gamma$. Then there is a constant C independent of f such that*

$$\left\|\tilde{D}^{\alpha}f\right\|_{\mathrm{Lip}(\beta-\alpha)}\leq C\left\|f\right\|_{\mathrm{Lip}(\beta)}.$$

Proof. We will prove the case when d is a metric. The proof for normalized spaces of homogeneous type has appeared in [**GSV**].

Let $f\in\mathrm{Lip}(\beta)$. Note that$\tilde{D}^{\alpha}f$ converges absolutely for any x. Let $x_1\neq x_2$, $r=d\left(x_1,x_2\right)$ and $B=B_{2r}\left(x_2\right)$. We have

$$\begin{aligned}\tilde{D}^{\alpha}f\left(x_2\right)-\tilde{D}^{\alpha}f\left(x_1\right) &= \int_B\frac{f\left(y\right)-f\left(x_2\right)}{d^{\alpha+n}\left(x_2,y\right)}d\mu\left(y\right)-\int_B\frac{f\left(y\right)-f\left(x_1\right)}{d^{\alpha+n}\left(x_1,y\right)}d\mu\left(y\right)+\\ &\quad\int_{B^c}\frac{f\left(y\right)-f\left(x_2\right)}{d^{\alpha+n}\left(x_2,y\right)}-\frac{f\left(y\right)-f\left(x_1\right)}{d^{\alpha+n}\left(x_1,y\right)}d\mu\left(y\right)\\ &= I_1+I_2+I_3.\end{aligned}$$

Furthermore, the last integral can be rewritten

$$\begin{aligned}I_3 &= \int_{B^c}\frac{f\left(x_1\right)-f\left(x_2\right)}{d^{\alpha+n}\left(x_2,y\right)}+\frac{f\left(y\right)-f\left(x_1\right)}{d^{\alpha+n}\left(x_2,y\right)}-\frac{f\left(y\right)-f\left(x_1\right)}{d^{\alpha+n}\left(x_1,y\right)}d\mu\left(y\right)\\ &= \int_{B^c}\frac{f\left(x_1\right)-f\left(x_2\right)}{d^{\alpha+n}\left(x_2,y\right)}d\mu\left(y\right)+\\ &\quad\int_{B^c}f\left(y\right)-f\left(x_1\right)\left\{\frac{1}{d^{\alpha+n}\left(x_2,y\right)}-\frac{1}{d^{\alpha+n}\left(x_1,y\right)}\right\}d\mu\left(y\right)\\ &= J_1+J_2.\end{aligned}$$

Now observe that

$$\begin{aligned}\left|I_1\right| &\leq \int_{d(x_2,y)\leq 2d(x_1,x_2)}\frac{\left\|f\right\|_{\mathrm{Lip}(\beta)}}{d^{n+\alpha-\beta}\left(x_2,y\right)}d\mu\left(y\right)\\ &\leq c\left\|f\right\|_{\mathrm{Lip}(\beta)}d^{\beta-\alpha}\left(x_1,x_2\right)\end{aligned}$$

and similarly

$$\begin{aligned}\left|I_2\right| &\leq \int_{d(x_1,y)\leq 3d(x_1,x_2)}\frac{\left\|f\right\|_{\mathrm{Lip}(\beta)}}{d^{n+\alpha-\beta}\left(x_1,y\right)}d\mu\left(y\right)\\ &\leq c\left\|f\right\|_{\mathrm{Lip}(\beta)}d^{\beta-\alpha}\left(x_1,x_2\right).\end{aligned}$$

On the other hand

$$\begin{aligned}\left|J_1\right| &\leq \left\|f\right\|_{\mathrm{Lip}(\beta)}d^{\beta}\left(x_1,x_2\right)\int_{d(x_2,y)>2d(x_1,x_2)}\frac{1}{d^{\alpha+n}\left(x_2,y\right)}d\mu\left(y\right)\\ &\leq c\left\|f\right\|_{\mathrm{Lip}(\beta)}d^{\beta-\alpha}\left(x_1,x_2\right)\end{aligned}$$

and

$$\begin{aligned}\left|J_2\right| &\leq d\left(x_1,x_2\right)\int_{d(x_2,y)>2d(x_1,x_2)}\frac{c\left\|f\right\|_{\mathrm{Lip}(\beta)}}{d^{n+\alpha+1-\beta}\left(x_2,y\right)}d\mu\left(y\right)\\ &\leq c\left\|f\right\|_{\mathrm{Lip}(\beta)}d^{\beta-\alpha}\left(x_1,x_2\right).\end{aligned}$$

Then

$$\left|I_3\right|\leq c\left\|f\right\|_{\mathrm{Lip}(\beta)}d^{\beta-\alpha}\left(x_1,x_2\right)$$

and finally

$$\left|\tilde{D}^{\alpha} f\left(x_{2}\right)-\tilde{D}^{\alpha} f\left(x_{1}\right)\right| \leq c\|f\|_{\operatorname{Lip}(\beta)} d^{\beta-\alpha}\left(x_{1}, x_{2}\right),$$

which concludes the proof. □

We would like to consider next fractional integrals on Lipschitz spaces.

Let (X, d, μ) be a nonhomogeneous space or a normalized space of homogeneous type of order $\gamma, 0<\gamma \leq 1$. Note that $\gamma=1$ when d is a metric, and $n=1$ when d is a quasidistance. We define the fractional integral of order α, $0<\alpha<\gamma \leq 1$ of a function $f \in \operatorname{Lip}(\beta) \cap L^{1}$, $0<\alpha<\beta$, as

$$I_{\alpha} f(x)=\int_{X} \frac{1}{d^{n-\alpha}(x, y)} f(y) d \mu(y)$$

and for $f \in \operatorname{Lip}(\beta)$, as

$$\widetilde{I_{\alpha}}(x)=\int_{X}\left\{\frac{1}{d^{n-\alpha}(x, y)}-\frac{1}{d^{n-\alpha}\left(x_{0}, y\right)}\right\} f(y) d \mu(y)$$

where x_{0} is a fixed point in X.

Note that the last integral converges both locally and at ∞. Of course the function $\widetilde{I_{\alpha}} f$ depends on the choice of x_{0}. But the functions obtained for different choices of x_{0} differ only by a constant.

Theorem 3. *Let (X, d, μ) be a nonhomogeneous space or a normalized space of homogeneous space of order γ. Let $\alpha, \beta>0$ be such that $\alpha+\beta<\gamma \leq 1$. Then $\widetilde{I_{\alpha}}$ is a bounded operator from* $\operatorname{Lip}(\beta)$ *to* $\operatorname{Lip}(\alpha+\beta)$ *if and only if $\widetilde{I_{\alpha}}(1)(x)=0$, for all x.*

Proof. We will prove the case when d is a metric. The proof for normalized spaces of homogeneous type has appeared in [**GV**]. Note again, that $\gamma=1$ when d is metric and $0<\alpha<1$. To see that the condition is necessary, observe that the continuity of the operator $\widetilde{I_{\alpha}}$ implies that $\widetilde{I_{\alpha}}(1)$ must be constant. On the other hand $\widetilde{I_{\alpha}}(1)\left(x_{0}\right)=0$, therefore the constant has to be 0.

To prove the sufficiency we consider $x \neq y$ points of X.

Since $\widetilde{I_{\alpha}}(1)=0$, observe first that

$$\widetilde{I_{\alpha}}(1)(x)-\widetilde{I_{\alpha}}(1)(y)=0 \Longleftrightarrow \int_{X}\left\{\frac{1}{d^{n-\alpha}(x, z)}-\frac{1}{d^{n-\alpha}(y, z)}\right\} d \mu(z)=0,$$

where the integral above converges because $0<\alpha<1$.

Thus we can write

$$\begin{aligned} & \widetilde{I_{\alpha}}(f)(x)-\widetilde{I_{\alpha}}(f)(y) \\ = & \int_{X}\left\{\frac{1}{d^{n-\alpha}(x, z)}-\frac{1}{d^{n-\alpha}(y, z)}\right\}\{f(z)-f(x)\} d \mu(z) \\ = & I+I I \end{aligned}$$

where I is the integral over $2B$, B being the ball with center x and radius $r = d(x,y)$ and II is the integral over $X \backslash 2B$. Now

$$|I| \leq \int_{2B} \frac{1}{d(x,z)^{n-\alpha}} |f(z) - f(x)| \, d\mu(z) + \int_{2B} \frac{1}{d(y,z)^{n-\alpha}} |f(z) - f(x)| \, d\mu(z).$$

In the sum above, both terms can be estimated in the same fashion.

For the first one, using Lemma 1, we get

$$\int_{2B} \frac{d(x,z)^{\beta}}{d(x,z)^{n-\alpha}} d\mu(z) \leq C(2r)^{\alpha+\beta};$$

and for the second one, enlarging $2B$ to the ball $B(y,3r)$, we get the same estimate.

In order to estimate II, we use Lemma 2 and Lemma 1 to obtain

$$\begin{aligned} II &\leq C \int_{X\backslash 2B} \frac{d(x,y)\, d(x,z)^{\beta}}{d(x,z)^{n-\alpha+1}} d\mu(z) \leq C d(x,y) \int_{X\backslash 2B} \frac{d\mu(z)}{d(x,z)^{n+1-\alpha-\beta}} \\ &\leq C d(x,y)\, r^{\alpha+\beta-1} \leq C d(x,y)^{\alpha+\beta}. \end{aligned}$$

This finishes the proof. □

4. On the composition of a fractional derivative and a fractional integral of the same order

The main result of this section is that the kernel of the composition of a fractional derivative and a fractional integral of the same order is a singular integral kernel that satisfies standard conditions.

In this section (X, d, μ) will denote a metric space with a non-doubling n-dimensional measure μ. i. e., $\mu(B_r(x)) \leq c_n r^n$ for some $n > 0$ with c_n independent of x and r.

The result for normalized spaces of homogeneous type was obtained in [**GSV**]. The proof that we present here follows that one. The fact that d is a metric makes the proof somewhat simpler.

We now define a singular integral kernel: Let $\Omega = X \times X \backslash \Delta$ where $\Delta = \{(x,y) : x = y\}$. A function $K(x,y) : \Omega \to \mathbb{C}$ is called a standard n-dimensional singular kernel when there are constants η, $0 < \eta < 1$, $\nu > 1$ and $M = M_{\nu,\eta} > 0$ such that

$$|K(x,y)| \leq \frac{M}{d^n(x,y)}, \tag{4.1}$$

and, for $\nu d(x,y) < d(x,z)$ we have

$$|K(x,z) - K(y,z)| \leq M \frac{d^{\eta}(x,y)}{d^{n+\eta}(x,z)} \tag{4.2}$$

$$|K(z,x) - K(z,y)| \leq M \frac{d^{\eta}(x,y)}{d^{n+\eta}(x,z)}. \tag{4.3}$$

THEOREM 4. *Let $0 < \alpha < 1$. Then $T_\alpha = D_\alpha I_\alpha$ is a singular integral operator with associated kernel*

$$K(x,y) = \int_X \frac{1}{d^{n+\alpha}(x,t)} \left[\frac{1}{d^{n-\alpha}(x,z)} - \frac{1}{d^{n-\alpha}(y,z)} \right] d\mu(t).$$

PROOF. We will show first that $K(x,y)$ satisfies the standard conditions (4.1), (4.2), and (4.3). To prove (4.1) i.e. $|K(x,y)| \leq \frac{M}{d^n(x,y)}$, $x \neq y$, observe that $|K(x,y)| \leq \Psi(x,y)$ where $\Psi(x,y) = \int_X \frac{1}{d^{n+\alpha}(x,t)} \left| \frac{1}{d^{n-\alpha}(x,z)} - \frac{1}{d^{n-\alpha}(y,z)} \right| d\mu(t)$. For fixed $x \neq y$ we break up X into three regions:

$$\begin{aligned} D_1 &= \{t : 2d(x,y) \leq d(x,t)\}, \\ D_2 &= \left\{t : \frac{1}{2} d(x,y) \leq d(x,t) \leq 2d(x,y)\right\}, \text{ and} \\ D_3 &= \left\{t : d(x,t) < \frac{1}{2} d(x,y)\right\}. \end{aligned}$$

On D_1, $d(t,y) \geq d(t,x) - d(x,y) \geq d(x,y)$, therefore

$$\Psi(x,y) \leq 2 \int_{D_1} \frac{1}{d^{n+\alpha}(x,t)} \cdot \frac{1}{d^{n-\alpha}(x,y)} d\mu(t) \leq \frac{M_1}{d^n(x,y)}.$$

On D_2, $d(y,t) \leq d(y,x) + d(x,t) \leq 3d(x,y)$; therefore

$$\begin{aligned} \int_{D_2} \frac{1}{d^{n+\alpha}(x,t)} \left| \frac{1}{d^{n-\alpha}(t,y)} \right| d\mu(t) &\leq \frac{1}{d^{n+\alpha}(x,y)} \int_{d(y,t) \leq 3d(x,y)} \frac{c_\alpha}{d^{n-\alpha}(t,y)} d\mu(t) \\ &\leq \frac{M_2}{d^n(x,y)}; \end{aligned}$$

and

$$\begin{aligned} \int_{D_2} \frac{1}{d^{n+\alpha}(x,t)} \left| \frac{1}{d^{n-\alpha}(t,y)} \right| d\mu(t) &\leq \frac{c_\alpha}{d^{n-\alpha}(x,y)} \int_{\frac{1}{2}d(x,y) < d(x,t)} \frac{1}{d^{n+\alpha}(x,t)} d\mu(t) \\ &\leq \frac{M_3}{d^n(x,y)}. \end{aligned}$$

Finally, on D_3, we have

$$\begin{aligned} \int_{D_3} \frac{1}{d^{n+\alpha}(x,t)} \left| \frac{1}{d^{n-\alpha}(t,y)} - \frac{1}{d^{n-\alpha}(x,y)} \right| d\mu(t) &\leq \int_{D_3} \frac{c_\alpha}{d^{n+\alpha}(x,t)} \frac{d(t,x)}{d^{n-\alpha+1}(x,t)} d\mu(t) \\ &\leq \frac{M_4}{d^n(x,y)}. \end{aligned}$$

We will consider now (4.2). Let x, y, z be fixed points satisfying

$$8d(x,y) \leq d(x,z).$$

Observe that

$$|K(x,z) - K(y,z)| \leq \tag{4.4}$$

$$\int \left| \frac{1}{d^{n+\alpha}(x,t)} \left(\frac{1}{d^{n-\alpha}(t,z)} - \frac{1}{d^{n-\alpha}(x,z)} \right) - \frac{1}{d^{n+\alpha}(y,t)} \left(\frac{1}{d^{n-\alpha}(t,z)} - \frac{1}{d^{n-\alpha}(y,z)} \right) \right| d\mu(t).$$

We now divide X into two regions, $A = \left\{t : \frac{1}{2} d(x,z) < d(x,t)\right\}$ and its complement A^c.

To estimate the integral (4.4) on A we rewrite the integrand as follows:

$$\left| \left(\frac{1}{d^{n+\alpha}(x,t)} - \frac{1}{d^{n+\alpha}(y,t)} \right) \frac{1}{d^{n-\alpha}(t,z)} + \frac{1}{d^{n+\alpha}(y,t)} \left(\frac{1}{d^{n-\alpha}(y,z)} - \frac{1}{d^{n-\alpha}(x,z)} \right) + \left(\frac{1}{d^{n+\alpha}(y,t)} - \frac{1}{d^{n+\alpha}(x,t)} \right) \frac{1}{d^{n-\alpha}(y,z)} \right| = |I_1 + I_2 + I_3|.$$

We estimate first $\int_A |I_3|\, d\mu(t)$. Observe that for $t \in A$, $4d(x,y) < d(x,t)$; therefore applying lemma 2 we have

$$\begin{aligned} \int_A |I_3|\, d\mu(t) &\leq c\frac{d(x,y)}{d^{n-\alpha}(y,z)} \int_A \frac{1}{d^{n+\alpha+1}(x,t)} d\mu(t) \\ &\leq c\frac{d(x,y)}{d^{n-\alpha}(y,z)} d^{-\alpha-1}(x,z) \leq c\frac{d(x,y)}{d^{n+1}(x,z)} \end{aligned}$$

because $d(y,z) \geq d(z,x) - d(x,y) \geq \frac{7}{8} d(x,z)$.

To estimate $\int_A |I_2|\, d\mu(t)$, observe that

$$|I_2| \leq \frac{1}{d^{n+\alpha}(y,t)} \cdot \frac{d(x,y)}{d^{n-\alpha+1}(x,z)} \leq c\frac{d(x,y)}{d^{n-\alpha+1}(x,z)} \cdot \frac{1}{d^{n+\alpha}(x,t)}$$

because $d(y,t) \geq \frac{3}{4} d(x,t)$. Then upon integrating over A we have

$$\int_A |I_2|\, d\mu(t) \leq c\frac{d(x,y)}{d^{n-\alpha+1}(x,z)} d^{-\alpha}(x,z) \leq c\frac{d(x,y)}{d^{n+1}(x,z)}$$

To estimate $\int_A |I_1|\, d\mu(t)$ we will further subdivide A into

$$\begin{aligned} D_1 &= \{t : d(x,t) > 2d(x,z)\}, \text{ and} \\ D_2 &= \left\{t : \frac{1}{2} d(x,z) < d(x,t) \leq 2d(x,z)\right\}. \end{aligned}$$

For $t \in D_1$, $d(x,t) > 2d(x,z) > 16d(x,y)$; therefore

$$\int_{D_1} |I_1|\, d\mu(t) \leq cd(x,y) \int_{D_1} \frac{1}{d^{n+\alpha+1}(x,t)} \frac{1}{d^{n-\alpha}(t,z)} d\mu(t).$$

Note now that for $t \in D_1$, $\frac{1}{2} d(x,t) \leq d(z,t)$, then

$$\int_{D_1} |I_1|\, d\mu(t) \leq c\frac{d(x,y)}{d^{n+1}(x,z)}.$$

To estimate $\int_{D_2} |I_1|\, d\mu(t)$, observe that for $t \in D_2$, $8d(x,y) < d(x,z) < 2d(x,t)$, then

$$|I_1| \leq \frac{cd(x,y)}{d^{n+\alpha+1}(x,t)} \cdot \frac{1}{d^{n-\alpha}(t,z)} \leq c\frac{d(x,y)}{d^{n+\alpha+1}(x,z)} \cdot \frac{1}{d^{n-\alpha}(t,z)},$$

and since D_2 is contained in the ball $\{t : d(t,z) \leq d(t,x) + d(x,z) \leq 3d(x,z)\}$, we have

$$\int_{D_2} |I_1|\, d\mu(t) \leq c\frac{d(x,y)}{d^{n+\alpha+1}(x,z)} \int_{d(t,z) \leq 3d(x,z)} \frac{1}{d^{n-\alpha}(t,z)} d\mu(t) \leq \frac{cd(x,y)}{d^{n+1}(x,z)}.$$

Now we estimate the integral in (4.4) on $A^c = \{t : 2d(x,t) < d(x,z)\}$. We divide this region into two subregions

$$\begin{aligned} B_1 &= \{t : d(x,t) < 2d(x,y)\}, \text{ and} \\ B_2 &= A^c \backslash B_1 = \left\{t : 2d(x,y) \leq d(x,t) \leq \frac{1}{2} d(x,z)\right\}. \end{aligned}$$

To estimate (4.4) on B_1, observe that $d(x,t) < 2d(x,y) < \frac{1}{4} d(x,z)$ and that $d(y,t) \leq d(y,x) + d(x,t) \leq \frac{1}{8} d(x,z) + \frac{1}{2} d(x,z) \leq \frac{5}{8} d(x,z) \leq \frac{5}{8}\frac{8}{7} d(y,z) = \frac{5}{7} d(y,z)$.

Therefore we can majorize the integral of (4.4) on B_1 by

$$\int_{B_1} \frac{1}{d^{n+\alpha}(x,t)} \left| \frac{1}{d^{n-\alpha}(t,z)} - \frac{1}{d^{n-\alpha}(x,z)} \right| d\mu(t) + \int_{B_1} \frac{1}{d^{n+\alpha}(y,t)} \left| \frac{1}{d^{n-\alpha}(t,z)} - \frac{1}{d^{n-\alpha}(y,z)} \right| d\mu(t).$$

The first term is less than or equal to

$$\int_{B_1} \frac{1}{d^{n+\alpha}(x,t)} \frac{d(t,x)}{d^{n-\alpha+1}(x,z)} d\mu(t) \le c \frac{d^{(1-\alpha)}(x,y)}{d^{n+(1-\alpha)}(x,z)}.$$

The second term is less than or equal to

$$\int_{d(y,t)\le 3d(x,y)} \frac{1}{d^{n+\alpha}(y,t)} \frac{d(t,y)}{d^{n-\alpha+1}(y,z)} d\mu(t) \le c \frac{d^{(1-\alpha)}(x,y)}{d^{n+(1-\alpha)}(y,z)} \le \frac{d^{(1-\alpha)}(x,y)}{d^{n+(1-\alpha)}(x,z)}$$

because $\frac{7}{8} d(x,z) \le d(y,z)$.

To estimate the integral (4.4) on B_2, we majorize it by

$$\int_{B_2} \left| \frac{1}{d^{n+\alpha}(x,t)} - \frac{1}{d^{n+\alpha}(y,t)} \right| \left| \frac{1}{d^{n-\alpha}(t,z)} - \frac{1}{d^{n-\alpha}(x,z)} \right| d\mu(t) + \int_{B_2} \frac{1}{d^{n+\alpha}(y,t)} \left| \frac{1}{d^{n-\alpha}(y,z)} - \frac{1}{d^{n-\alpha}(x,z)} \right| d\mu(t) = J_1 + J_2.$$

To estimate J_1 observe that $2d(x,y) \le d(x,t)$ and that $2d(x,t) \le d(x,z)$; then J_1 is less than or equal to

$$\begin{aligned} C \int_{B_2} \frac{d(x,y)}{d^{n+\alpha+1}(x,t)} \frac{d(t,x)}{d^{n-\alpha+1}(x,z)} d\mu(t) &\le \\ C \frac{d(x,y)}{d^{n-\alpha+1}(x,z)} \int_{2d(x,y)\le d(x,t)} \frac{1}{d^{n+\alpha}(x,t)} d\mu(t) &\le C \frac{d^{(1-\alpha)}(x,y)}{d^{n+(1-\alpha)}(x,z)}. \end{aligned}$$

Finally note that

$$\begin{aligned} J_2 &\le C \int_{B_2} \frac{1}{d^{n+\alpha}(y,t)} \frac{d(y,x)}{d^{n-\alpha+1}(x,z)} d\mu(t) \\ &\le C \frac{d(y,x)}{d^{n-\alpha+1}(x,z)} \int_{d(x,y)\le d(y,t)} \frac{1}{d^{n+\alpha}(y,t)} d\mu(t) \le C \frac{d^{(1-\alpha)}(y,x)}{d^{n+(1-\alpha)}(x,z)}. \end{aligned}$$

This concludes the proof of (4.2) with $\nu = 8$ and $\eta = 1 - \alpha$.

Let x, y, z be fixed points such that $8d(x,y) < d(x,z)$ and $0 < \alpha < 1$. We have

$$|K(z,x) - K(z,y)| = \left| \int \frac{1}{d^{n+\alpha}(z,t)} \left[\left(\frac{1}{d^{n-\alpha}(t,x)} - \frac{1}{d^{n-\alpha}(z,x)} \right) - \left(\frac{1}{d^{n-\alpha}(t,y)} - \frac{1}{d^{n-\alpha}(z,y)} \right) \right] d\mu(t) \right|. \tag{4.5}$$

To estimate this integral we divide X into three regions:

$$\begin{aligned} D &= \{t : 2d(z,t) < \min\{d(y,z), d(x,z)\}\} \\ E &= \left\{ t : \frac{1}{2} \min\{d(y,z), d(x,z)\} \le d(z,t) \le 2d(x,z) \right\} \\ F &= \{t : 2d(x,z) \le d(z,t)\}. \end{aligned}$$

To estimate the integral (4.5) on D we further subdivide D into two subregions: $D_1 = \{t : d(z,t) \le 2d(x,y)\}$ and $D_2 = D \backslash D_1$.

The integral (4.5) on D_1 is less than or equal to

$$\int_{D_1} \frac{1}{d^{n+\alpha}(z,t)} \left| \frac{1}{d^{n-\alpha}(t,x)} - \frac{1}{d^{n-\alpha}(z,x)} \right| d\mu(t) + \int_{D_1} \frac{1}{d^{n+\alpha}(z,t)} \left| \frac{1}{d^{n-\alpha}(t,y)} - \frac{1}{d^{n-\alpha}(z,y)} \right| d\mu(t). \tag{4.6}$$

Since $2d(z,t) \leq d(x,z)$ on D_1, the first term is less than or equal to

$$\frac{c}{d^{n-\alpha+1}(z,y)} \int_{d(z,t)\leq 2d(x,y)} \frac{d(t,z)}{d^{n+\alpha}(t,z)} d\mu(t) \leq c\frac{d^{(1-\alpha)}(x,y)}{d^{n+(1-\alpha)}(x,z)};$$

and since $2d(z,t) \leq d(y,z)$ and $\frac{7}{8}d(z,x) \leq d(z,y)$, the second term of (4.6) is less than or equal to

$$\frac{c}{d^{n-\alpha+1}(z,y)} \int_{d(z,t)\leq 2d(x,y)} \frac{d(t,z)}{d^{n+\alpha}(t,z)} d\mu(t) \leq c\frac{d^{(1-\alpha)}(x,y)}{d^{n+(1-\alpha)}(x,z)}.$$

Let's consider now integral (4.5) on D_2. Here we have $2d(x,y) < d(z,t) < \frac{1}{2}\min\{d(y,z), d(x,z)\}$. This integral is less than or equal to

$$\int_{D_2} \frac{1}{d^{n+\alpha}(z,t)} \left| \frac{1}{d^{n-\alpha}(t,x)} - \frac{1}{d^{n-\alpha}(t,y)} \right| d\mu(t) + \int_{D_2} \frac{1}{d^{n+\alpha}(z,t)} \left| \frac{1}{d^{n-\alpha}(z,y)} - \frac{1}{d^{n-\alpha}(z,x)} \right| d\mu(t). \tag{4.7}$$

To estimate the first term,observe that $d(x,t) \geq d(x,z) - d(z,t) \geq d(z,t) > d(x,y)$; therefore the first integral of (4.7) is less than or equal to

$$\int_{D_2} \frac{1}{d^{n+\alpha}(z,t)} \frac{c}{d^{n-\alpha+1}(x,t)} d\mu(t) \leq \frac{cd(x,y)}{d^{n-\alpha+1}(x,z)} \int_{2d(x,y)<d(z,t)} \frac{1}{d^{n+\alpha}(z,t)} d\mu(t)$$

because $d(x,t) \geq \frac{1}{2}d(x,z)$, and the last expression is less than or equal

$$c\frac{d^{(1-\alpha)}(x,y)}{d^{n+(1-\alpha)}(x,z)}.$$

The second term of (4.7) is less than or equal to

$$c\frac{d(y,x)}{d^{n-\alpha+1}(x,z)} \int_{2d(x,y)<d(z,t)} \frac{1}{d^{n+\alpha}(z,t)} d\mu(t) \leq c\frac{d^{(1-\alpha)}(x,y)}{d^{n+(1-\alpha)}(x,z)}.$$

The integral (4.5) over E is less than or equal to

$$\int_{E} \frac{1}{d^{n+\alpha}(z,t)} \left| \frac{1}{d^{n-\alpha}(t,x)} - \frac{1}{d^{n-\alpha}(t,y)} \right| d\mu(t) + \int_{E} \frac{1}{d^{n+\alpha}(z,t)} \left| \frac{1}{d^{n-\alpha}(z,y)} - \frac{1}{d^{n-\alpha}(z,x)} \right| d\mu(t). \tag{4.8}$$

To estimate the second integral observe that $8d(x,y) < d(z,x)$ and that $\frac{7}{8}d(x,z) \leq d(z,y)$; therefore it is less than or equal to

$$\frac{c\,d(x,y)}{d^{n-\alpha+1}(z,x)} \int_{\frac{7}{16}d(x,z)\leq d(z,t)} \frac{1}{d^{n+\alpha}(z,t)} d\mu(t) \leq c\frac{d(x,y)}{d^{n+1}(z,x)}.$$

To estimate the first integral of (4.8) we further subdivide E into two regions

$$\begin{aligned} E_1 &= \{t \in E : d(t,x) < 2d(x,y)\} \\ E_2 &= \{t \in E : d(t,x) \geq 2d(x,y)\}. \end{aligned}$$

Observe that $\frac{8}{9}d(z,y) \leq d(z,x) \leq \frac{8}{7}d(z,y)$; therefore the first integral on E_1 is less than or equal to

$$\frac{c}{d^{n+\alpha}(x,z)} \int_{d(t,x)<2d(x,y)} \frac{1}{d^{n-\alpha}(t,x)} d\mu(t) + \frac{c}{d^{n+\alpha}(x,z)} \int_{d(t,y)<3d(x,y)} \frac{1}{d^{n-\alpha}(t,y)} d\mu(t) \leq c\frac{d^{\alpha}(x,y)}{d^{n+\alpha}(x,z)}.$$

For $t \in E_2$, the first integral of (4.8) is majorized by

$$\int_{E_2} \frac{1}{d^{n+\alpha}(z,t)} \frac{d(x,y)}{d^{n-\alpha+1}(x,t)} d\mu(t) \leq \frac{c\, d(x,y)}{d^{n+\alpha}(x,z)} \int_{d(x,y)\geq 2d(x,y)} \frac{1}{d^{n+(1-\alpha)}(x,t)} d\mu(t)$$

because $\frac{7}{16}d(z,x) \leq d(z,t)$; and the last expression is less than or equal to

$$c\frac{d^{\alpha}(x,y)}{d^{n+\alpha}(x,z)}.$$

Finally we will estimate the integral (4.5) over F. This integral is less than or equal to

$$\int_F \frac{1}{d^{n+\alpha}(z,t)} \left| \frac{1}{d^{n-\alpha}(t,x)} - \frac{1}{d^{n-\alpha}(t,x)} \right| d\mu(t) + \int_F \frac{1}{d^{n+\alpha}(z,t)} \left| \frac{1}{d^{n-\alpha}(z,y)} - \frac{1}{d^{n-\alpha}(z,x)} \right| d\mu(t). \tag{4.9}$$

To estimate the first integral in (4.9), observe that for $t \in F$, $d(x,z) \leq d(x,t)$. Therefore, this integral is less than or equal to

$$\begin{aligned} c\, d(x,y) \int_F \frac{d\mu(t)}{d^{n+\alpha}(t,z)\, d^{n-\alpha}(t,x)} &\leq c\frac{d(x,y)}{d^{n-\alpha+1}(x,z)} \int_{2d(x,z)<d(z,t)} \frac{d\mu(t)}{d^{n+\alpha}(t,z)} \\ &\leq c\frac{d(x,y)}{d^{n+1}(x,z)}. \end{aligned}$$

To estimate the second integral in (4.9), note that $8d(x,y) \leq d(x,z)$; then this integral is less than or equal to

$$\frac{d(x,y)}{d^{n-\alpha+1}(x,z)} \int_{2d(x,z)\leq d(z,t)} \frac{d\mu(t)}{d^{n+\alpha}(z,t)} \leq c\frac{d(x,y)}{d^{n+1}(x,z)}.$$

This concludes the proof of the standard estimates with $\eta = \min\{\alpha, 1-\alpha\}$.

It remains to show that T_α is associated with the kernel K, i. e., that

$$T_\alpha(x) = \int K(x,y) f(y) d\mu(y)$$

for μ-a.e. $x \in \operatorname{supp}(\mu) \backslash \operatorname{supp}(f)$, $f \in \operatorname{Lip}(\beta), \alpha+\beta < \dot{1}$. Let $f \in \operatorname{Lip}(\beta), \alpha+\beta < 1$, then

$$\begin{aligned} D_\alpha I_\alpha f(x) &= \int \frac{(I_\alpha f)(t) - (I_\alpha f)(x)}{d^{n+\alpha}(x,t)} d\mu(t) \\ &= \int_E \frac{1}{d^{n+\alpha}(x,t)} \left\{ \int \left[\frac{1}{d^{n-\alpha}(t,y)} - \frac{1}{d^{n-\alpha}(x,y)} \right] f(y) d\mu(y) \right\} d\mu(t). \end{aligned}$$

For $x \in \operatorname{supp}(\mu) \backslash \operatorname{supp}(f)$, using the estimate obtained above for $\Psi(x,y)$, it can be seen that the last integral converges absolutely. Changing the order of integration, we have

$$D_\alpha I_\alpha f(x) = \int K(x,y) f(y) d\mu(y) = T_\alpha(x).$$

□

From this point and until the end of the paper we will only consider the case of a normalized space of homogeneous type of order γ. These results were obtained in [GSV] and we will reproduce them here. The corresponding results for non-homogeneous spaces are in progress and will appear elsewhere.

5. Construction of an equivalent quasidistance with the cancellation property $\tilde{I}_\alpha 1 = 0$

The first lemma states the properties of a Coifman type approximation to the identity. These properties are well known,see [DJS], and therefore the proofs will be omitted.

Let $h \geq 0$ be a C^∞ function on $[0,\infty)$ such that $h(r) = 1$ for $0 \leq r \leq \frac{1}{2}$, and $h(r) = 0$ for $r \geq 2$. For $f \in L^1_{loc(X)}$ and $t > 0$ set

$$T_t f(x) = \frac{1}{t} \int_X h\left(\frac{\delta(x,y)}{t}\right) f(y) d\mu(y),$$

$$M_t f(x) = \frac{1}{(T_t 1)(x)} f(x) = \varphi(x,t) f(x),$$

$$V_t f(x) = \frac{1}{T_t(\frac{1}{T_t 1})(x)} f(x) = \psi(x,t) f(x).$$

Now define S_t by

$$S_t = M_t T_t V_t T_t M_t,$$

then

$$S_t f(x) = \int_X s(x,y,t) f(y) d\mu(y),$$

where

$$s(x,y,t) = \frac{\varphi(x,t)\varphi(y,t)}{t^2} \int_X h\left(\frac{\delta(x,u)}{t}\right) h\left(\frac{\delta(y,u)}{t}\right) \psi(u,t) d\mu(u).$$

LEMMA 3. *There exist positive constants $b_1, b_2, c_1, c_2,$ and c_3 independent of $x, y,$ and t such that*

(i): $s(x,y,t) = s(y,x,t)$ for all x, y in X and $t > 0$,
(ii): $\mid s(x,y,t) \mid \leq \frac{c_1}{t}$ for all x, y in X and $t > 0$,
$s(x,y,t) = 0$ if $\delta(x,y) > b_1 t$,and
$\frac{c_2}{t} < s(x,y,t)$ if $\delta(x,y) < b_2 t$
(iii): $\mid s(x,y,t) - s(x',y,t) \mid < c_3 \frac{\delta^\gamma(x,x')}{t^{1+\gamma}}$ for all x,x' and y in X,and $t > 0$.
(iv): $\int s(x,y,t) d\mu(y) = 1$ for all x in X, and $t > 0$.

(v): $s(x,y,t)$ is continuously differentiable with respect to t.

LEMMA 4. *For each* α, $-\infty < \alpha < 1$, *the function* δ_α, *defined in (5.1) below is a quasidistance equivalent to* δ *and it satisfies (1.2). We define* $\delta_\alpha : X\times X \to [0,\infty)$ *by*

$$\delta_\alpha(x,y) = (\int_0^\infty t^{\alpha-1}s(x,y,t)dt)^{\frac{1}{\alpha-1}} \textit{ for } x\neq y \textit{ and} \tag{5.1}$$

$$\delta_\alpha(x,y) = 0 \textit{ for } x=y.$$

PROOF. We shall prove first that there are positive constants c'_α and c''_α such that for all x,y in X

$$c'_\alpha\delta(x,y) \leq \delta_\alpha(x,y) \leq c''_\alpha\delta(x,y). \tag{5.2}$$

Using the properties of $s(x,y,t)$ stated in Lemma 3, we have that $s(x,y,t)=0$ if $\delta(x,y) > b_1 t$. Then

$$\delta_\alpha^{\alpha-1}(x,y) = \int_{\frac{\delta(x,y)}{b_1}}^\infty t^{\alpha-1}s(x,y,t)dt. \tag{5.3}$$

On the other hand $\| s(\cdot,\cdot,t) \|_\infty \leq \frac{c_1}{t}$, and therefore

$$\delta_\alpha^{\alpha-1}(x,y) \leq c_1\int_{\frac{\delta(x,y)}{b_1}}^\infty t^{\alpha-2}dt = \frac{c_1}{1-\alpha}b_1^{1-\alpha}\delta^{\alpha-1}(x,y).$$

Raising this inequality to the power $1/(\alpha-1)$ we obtain the first inequality of (5.2).

To obtain the second inequality of (5.2) note that $s(x,y,t) \geq \frac{c_2}{t}$ if $\delta(x,y) < b_2 t$. Then

$$\delta_\alpha^{\alpha-1}(x,y) \geq \int_{\frac{\delta(x,y)}{b_2}}^\infty t^{\alpha-1}\frac{c_2}{t}dt = \frac{c_2}{1-\alpha}b_2^{1-\alpha}\delta^{\alpha-1}(x,y).$$

Raising this inequality to the power $\frac{1}{\alpha-1}$ we conclude the proof of (5.2).

The fact that $\delta_\alpha(x,y)$ is a quasidistance follows from the definition, property (i) of $s(x,y,t)$ and (5.2). We will denote by κ_α the constant in the triangle inequality for δ_α.

We will show now that δ_α satisfies (1.2). If $\delta_\alpha(x,y)=0$ then $x=y$ and $\delta_\alpha(x',y) = \delta_\alpha(x,x')$ and

$$\left|\delta_\alpha(x,y) - \delta_\alpha(x',y)\right| = \delta_\alpha(x,x') = \delta_\alpha^\gamma(x,x')\{\delta_\alpha(x,y)+\delta_\alpha(x',y)\}^{1-\gamma}.$$

Similarly when $\delta_\alpha(x',y)=0$ we get the estimate above.

Assume now that $\delta_\alpha(x,y)\neq 0$ and $\delta_\alpha(x',y)\neq 0$. Let $a = \frac{1}{b_1}\min\{\delta_\alpha(x,y),\delta_\alpha(x',y)\}$,then by property (ii) of Lemma 3

$$\left|\delta_\alpha(x,y) - \delta_\alpha(x',y)\right| = \left|(\int_a^\infty t^{\alpha-1}s(x,y,t)dt)^{\frac{1}{\alpha-1}} - (\int_a^\infty t^{\alpha-1}s(x',y,t)dt)^{\frac{1}{\alpha-1}}\right| \leq$$

$$\left[\int_a^\infty t^{\alpha-1}\left|s(x',y,t)+\theta(s(x,y,t)-s(x',y,t))\right|dt\right]^{\frac{2-\alpha}{\alpha-1}}\left[\int_a^\infty t^{\alpha-1}\left|s(x,y,t)-s(x',y,t)\right|dt\right],$$

with $0<\theta<1$. Using (ii) and (iii) of Lemma 3 we can majorize the last estimate by

$$\left(c\int_a^\infty t^{\alpha-2}dt\right)^{\frac{2-\alpha}{\alpha-1}}\left(\int_a^\infty t^{\alpha-\gamma-2}c\delta_\alpha^\gamma(x,x')dt\right)\leq$$

$$c\delta_\alpha^\gamma(x,x')a^{1-\gamma}\leq c\delta_\alpha^\gamma(x,x')\{\delta_\alpha(x,y)+\delta_\alpha(x',y)\}^{1-\gamma}$$

This concludes the proof of the lemma.

Lemma 5. *(Cancellation property). Let $0<\alpha<\gamma$, then*

$$\int_X[\delta_\alpha^{\alpha-1}(x,y)-\delta_\alpha^{\alpha-1}(x',y)]d\mu(y)=0,$$

for any x, x' in X.

Proof. We show first that

$$\int_X\int_0^\infty t^{\alpha-1}\mid s(x,y,t)-s(x'y,t)\mid d\mu(y)dt<\infty.$$

We have

$$\int_X\int_0^1 t^{\alpha-1}\mid s(x,y,t)-s(x'y,t)\mid d\mu(y)dt\leq\int_0^1 2t^{\alpha-1}dt<\infty$$

To estimate $\int_X\int_1^\infty t^\alpha\mid s(x,y,t)-s(x',y,t)\mid d\mu(y)dt$, observe that the functions $s(x,\cdot,t)$ are supported in balls of radius b_1t, also by (iii) of Lemma 3 we have

$$\mid s(x,y,t)-s(x',y,t)\mid\leq c_3\frac{\delta^\gamma(x,x')}{t^{1+\gamma}}.$$

Therefore using normality the double integral is majorized by

$$\int_1^\infty t^{\alpha-1}\frac{\delta^\gamma(x,x')ct}{t^{1+\gamma}}dt\leq c\delta^\gamma(x,x')\int_1^\infty\frac{dt}{t^{1+\gamma-\alpha}}<\infty.$$

Since

$$\int_X\left[\delta_\alpha^{\alpha-1}(x,y)-\delta_\alpha^{\alpha-1}(x',y)\right]d\mu(y)=$$

$$\int_X\int_0^\infty t^{\alpha-1}\left[s(x,y,t)-s(x',y,t)\right]dtd\mu(y),$$

by changing the order of integration and using (v) of Lemma 3 we obtain that the integral is zero.

6. Boundedness of T_α in L^2_μ

In this section δ will be the quasidistance $\delta_\alpha, 0<\alpha<\gamma$, constructed in section 5. We will denote $\mathrm{Lip}_B(\eta) = \{f \in \mathrm{Lip}(\eta) \text{ and } \mathrm{supp}(f) \subset B\}$, and $\mathrm{Lip}_0(\eta) = \cup_B \mathrm{Lip}_B(\eta)$.

THEOREM 5. *The operator T_α is bounded in L^2_μ.*

PROOF. To prove the theorem we use the "$T1$-theorem". We recall that an operator $T : \mathrm{Lip}_0(\eta) \to (\mathrm{Lip}_0(\eta))'$ is weakly bounded if there exists a constant c such that

$$| \langle Tf, g\rangle | \leq c\mu(B)^{1+2\eta} \|f\|_\eta \|g\|_\eta \tag{6.1}$$

for every f, g in $\mathrm{Lip}_B(\eta)$ and for every ball B. We will show that

(i) T_α is weakly bounded
(ii) $T_\alpha 1 = 0$
(iii) ${}^tT_\alpha 1 = 0$.

To prove (i) we will show first the following estimate for $f \in \mathrm{Lip}_B(\eta)$

$$\|T_\alpha f\|_\infty \leq c\mu(B)^\eta \parallel f \parallel_\eta . \tag{6.2}$$

Consider $f \in \mathrm{Lip}_B(\eta), B = B_r(x_o)$. Observe that

$$|I_\alpha f(x)| \leq \left| \int \frac{f(y)}{\delta^{1-\alpha}(x,y)} d\mu(y) \right| \leq \left| \int_B \frac{f(y)}{\delta^{1-\alpha}(x,y)} d\mu(y)\right| \leq c \|f\|_\infty \mu^\alpha(B).$$

Now

$$|T_\alpha f(x)| \leq \int \frac{|(I_\alpha f)(t) - (I_\alpha f)(x)|}{\delta^{1+\alpha}(x,t)} d\mu(t) \leq$$

$$\int\limits_{\delta(x,t)<r} \frac{|I_\alpha f(t) - I_\alpha f(x)|}{\delta^{1+\alpha}(x,t)} d\mu(t) + \int\limits_{\delta(x,t)\geq r} \frac{|I_\alpha f(t) - I_\alpha f(x)|}{\delta^{1+\alpha}(x,t)} d\mu(t).$$

To estimate the first integral we use the fact that $|I_\alpha f(x) - I_\alpha f(x)| \leq c \parallel f \parallel_\eta \delta^{\eta+\alpha}(t,x)$ proved in Theorem 1, and then integrating this integral is less than or equal to $c \parallel f \parallel_\eta \mu(B)^\eta$. For the second integral we use the estimate for $I_\alpha f$ obtained above and integrating we obtain that this integral is less than or equal $c \parallel f \parallel_\infty$.Note that for $f \in \mathrm{Lip}_0(\eta)(B), \parallel f \parallel_\infty \leq c \parallel f \parallel_\eta \mu(B)^\eta$, this concludes the proof of (6.2). Let f and g be in $\mathrm{Lip}_0(\eta)(B)$, then

$$|\langle T_\alpha f, g\rangle| \leq \int_B |T_\alpha f(x)| \mid g(x) \mid d\mu(x) \leq$$

$$\leq \|T_\alpha f\|_\infty \|g\|_\infty \mu(B) \leq c\mu(B)^{1+2\eta} \|f\|_\eta \|g\|_\eta .$$

To prove (ii) we observe that the extension of T_α to $L^\infty \cap Lip(\eta)$ coincides with the operator $\widetilde{T}_\alpha = \widetilde{D}_\alpha \widetilde{I}_\alpha$. Since $\widetilde{I}_\alpha 1 = 0$ we have $T_\alpha 1 = \widetilde{T}_\alpha 1 = 0$. To prove (iii) we use that

$${}^tT_\alpha = I_\alpha D_\alpha.$$

In fact, let $S_\alpha = I_\alpha D_\alpha$ and consider f and g in $\mathrm{Lip}_0(\beta)$, $0 < \alpha + \beta \leq \gamma$. We want to show that

$$\langle T_\alpha f, g\rangle = \langle f, S_\alpha g\rangle . \tag{6.3}$$

We will show first that for $f \in L^\infty \cap Lip(\eta)$, $\alpha < \eta \leq \gamma$ and $g \in \mathrm{Lip}_0(\beta)$

$$\langle D_\alpha f, g\rangle = \langle f, D_\alpha g\rangle . \tag{6.4}$$

For every $f \in L^\infty \cap Lip(\eta)$, note that $\int \frac{|f(t)-f(x)|}{\delta^{1+\alpha}(x,t)} d\mu(t)$ is bounded as a function of x and therefore

$$\langle D_\alpha f, g\rangle = \int\int \frac{[f(t)-f(x)]}{\delta^{1+\alpha}(x,t)} g(x) d\mu(t) d\mu(x)$$

because the double integral above converges absolutely. Now rewrite the last integral as follows

$$\int\int \frac{[f(t)g(x)-f(x)g(t)]}{\delta^{1+\alpha}(x,t)} d\mu(t)d\mu(x) + \int\int \frac{[f(x)g(t)-f(x)g(x)]}{\delta^{1+\alpha}(x,t)} d\mu(t)d\mu(x).$$

The second integral converges absolutely since for $g \in \mathrm{Lip}_{B_r(x)}(\beta)$

$$\int \frac{|\ g(t)-g(x)\ |}{\delta^{1+\alpha}(x,t)} d\mu(t) \leq \frac{c}{1+\delta^{1+\alpha}(x_o,x)},$$

and it is equal to $(f, D_\alpha g)$. Finally observe that the first integral is absolutely convergent (since the second one is), since the integrand $H(x,t)$ satisfies $H(x,t) = -H(t,x)$, its value is equal to 0.

Now consider f and g in $\mathrm{Lip}_0(\beta)$. It was shown before (see Theorem 3 and (6.2)) that $I_\alpha f \in L^\infty \cap Lip(\eta)$, therefore

$$\langle D_\alpha I_\alpha f, g\rangle = \langle I_\alpha f, D_\alpha g\rangle =$$

$$= \int D_\alpha g(x)\ \frac{f(t)}{\delta^{1-\alpha}(x,t)} d\mu(t)\ d\mu(x).$$

Since $I_\alpha(|\ f\ |) \in L^\infty$ and $|\ D_\alpha g(x)\ | \leq \frac{c}{1+\delta^{1+\alpha}(x,x_o)}$, the double integral converges absolutely and by Fubini's theorem is equal to $\langle f, I_\alpha D_\alpha g\rangle$. Finally, since $D_\alpha 1 = 0$ we have $T_\alpha 1 = 0$. This concludes the proof of the Theorem. □

7. Representation Formulas

In this section we will use the quasidistances δ_α and $\delta_{-\alpha}$ constructed in Section 5 in the definitions of fractional integral and fractional derivative respectively. The function $s(x,y,t)$ introduced in Section 5 is continuously differentiable in t. Let

$$q(x,y,t) = t\frac{\partial}{\partial t} s(x,y,t)$$

and set

$$-Q_t f(x) = \int_X q(x,y,t) f(y) d\mu(y)\ . \tag{7.1}$$

LEMMA 6. *The kernel $q(x,y,t)$ defined in (2.16) has the following properties*

:

LEMMA 7. *(i)* $q(x,y,t) = q(y,x,t)$ *for all* x, y *in* X*, and* $t > 0$.
(ii) $q(x,y,t) = 0$ *if* $\delta(x,y) > b_1 t$,
(iii) $\mid q(x,y,t) \mid \leq \frac{c_4}{t}$ *for all* x, y *in* X*, and* $t > 0$.
(iv) $\mid q(x,y,t) - q(x',y,t) \mid \leq c_5 \frac{\delta^\gamma(x,x')}{t^{1+\gamma}}$ *for all* x, x', y *in* X *and* $t > 0$.
(v) $\int q(x,y,t)\ d\mu(y) = 0$ *for all* x *in* X *and* $t > 0$.

Lemma 6 is the continuous version of known results [N1], [DJS].

THEOREM 6. *If* $Q_t(f)$ *is the operator defined by equation 7.1, then the following representation formulas hold pointwise everywhere and in the weak sense.*

$$\alpha I_\alpha f = \int_0^\infty t^\alpha Q_t(f) \frac{dt}{t},\ \textit{for}\ f \in\ Lip(\beta) \cap L^1, 0 < \alpha,\ \alpha + \beta \leq \gamma, \tag{7.2}$$

and

$$-\alpha D_\alpha f = \int_0^\infty t^{-\alpha} Q_t(f) \frac{dt}{t},\ \textit{for}\ f \in\ Lip(\beta) \cap L^\infty,\ 0 < \alpha < \beta \leq \gamma. \tag{7.3}$$

To prove 7.2, observe that for $f \in Lip(\beta) \cap L^1$ the fractional integral converges absolutely for every x, therefore using (5.1) we have

$$\alpha\ I_\alpha f(x) = \alpha \int_X \int_0^\infty t^{\alpha-1} s(x,y,t) dt\ f(y)\ d\mu(y)$$

and the double integral converges absolutely for every x. Then by changing the order of integration we obtain.

$$\alpha\ I_\alpha f(x) = \alpha \int_0^\infty t^{\alpha-1} u(x,t) dt \tag{7.4}$$

where

$$u(x,t) = \int_X s(x,y,t)\ f(y)\ d\mu(y). \tag{7.5}$$

Since $\frac{\partial}{\partial t}\ s(x,y,t) = \frac{1}{t}\ q(x,y,t)$ and $q(x,y,t)$ has the properties (ii) and (iii) of Lemma 6, we can differentiate with respect to t under the integral sign of (7.5) to get

$$\frac{\partial}{\partial t} u(x,t) = \frac{1}{t} \upsilon(x,t) \tag{7.6}$$

where

$$\upsilon(x,t) = \int_X q(x,y,t)\ f(y)\ d\mu(y) = -Q_t f(x). \tag{7.7}$$

Now integrating the integral in (7.4) by parts, using (7.6) and the fact that $f \in Lip(\beta) \cap L^1$ we obtain

$$\alpha\ I_\alpha f(x) = \lim_{\varepsilon \to 0}\ \alpha \int_\varepsilon^{1/\varepsilon} t^{\alpha-1} u(x,t) dt = \lim_{\varepsilon \to 0} (t^\alpha u(x,t) \mid_\varepsilon^{1/\varepsilon})$$

$$-\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon} t^{a}\upsilon(x,t)\frac{dt}{t}=\int_0^{\infty} t^{a}Q_t(f)(x)\frac{dt}{t}.$$

To prove (7.3) observe that for $f \in Lip(\beta)\cap L^{\infty}$ the fractional derivative converges absolutely for every x, therefore using (5.1) we have

$$-\alpha\ D_{\alpha}f(x)=-\alpha \int\limits_X \int_0^{\infty} t^{-\alpha-1}s(x,y,t)dt[f(y)-f(x)]d\mu(y)$$

and the double integral converges absolutely. Then by changing the order of integration we have

$$-\alpha\ D_{\alpha}f(x)=-\alpha \int_0^{\infty} t^{-\alpha-1}[u(x,t)-f(x)]dt. \tag{7.8}$$

Now integrating the integral in (7.8) by parts, using (7.6), and the fact that $f \in Lip(\beta)\cap L^{\infty}$, $\alpha<\beta\leq\gamma$ we obtain

$$-\alpha\ D_{\alpha}f(x)=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon} t^{-\alpha-1}[u(x,t)-f(x)]dt=$$

$$\lim_{\varepsilon\to 0}(t^{-\alpha}[u(x,t)-f(x)]\ |_{\varepsilon}^{1/\varepsilon})-\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon} t^{-\alpha}\upsilon(x,t)\frac{dt}{t}$$

$$=\int_0^{\infty} t^{-\alpha}Q_t(f)(x)\frac{dt}{t}.$$

This concludes the proof of the Theorem.

8. A fundamental theorem for fractional calculus

In this section again, we will use the quasidistances δ_{α} and $\delta_{-\alpha}$ constructed in Section 5 in the definitions of fractional integral and fractional derivative respectively. The main result of this section is that T_{α} is invertible in L^p_{μ}.

LEMMA 8. *For positive r, s, t define a function $h_t(r,s)$ as follows*

$$h_t(s,r)=\begin{cases}\min\left\{t^{\gamma},\left(\frac{s}{r}\right)^{\frac{\gamma}{2}}\right\} & \text{if } \frac{s}{r}\leq 1 \text{ and } 0<t\leq 1\\ \min\left\{t^{\gamma},\left(\frac{s}{r}\right)^{-\frac{\gamma}{2}}\right\} & \text{if } \frac{s}{r}>1 \text{ and } 0<t\leq 1\end{cases}$$

$$h_t(s,r)=\begin{cases}\min\left\{t^{-\gamma},\left(\frac{s}{r}\right)^{\frac{\gamma}{2}}\right\} & \text{if } \frac{s}{r}\leq 1 \text{ and } t>1\\ \min\left\{t^{-\gamma},\left(\frac{s}{r}\right)^{-\frac{\gamma}{2}}\right\} & \text{if } \frac{s}{r}>1 \text{ and } t>t\end{cases}.$$

Then the operators Q_t introduced in (7.1) satisfy

$$\|Q_sQ_{ts}(Q_rQ_{tr})^{*}\|\leq h_t(s,r)$$

and

$$\|\ (Q_sQ_{ts})^{*}Q_rQ_{tr}\ \|\ \ \leq h_t(s,r)$$

where the $\|\ \|$ is the operator norm on L^2_{μ}.

Moreover for $f\in L^2_{\mu}$ we have

$$\left\|\int_{\frac{1}{M}}^{M} Q_sQ_{ts}f\frac{ds}{s}\right\|_2\leq[\sup_s\int_0^{\infty} h_t(s,r)\frac{dr}{r}]\ \|\ f\ \|_2$$

uniformly in M, where $\| \|_2$is the L^2_μ norm.

These are continuous versions of known results, see for example [DJS] and [N1], the last inequality follows from the continuous version of the Cotlar-Knapp-Stein lemma.

LEMMA 9. *For positive t let*

$$c_t = \sup_s \int_0^\infty h_t(s,r)\frac{dr}{r}$$

then

$$c_t \leq c \begin{cases} t^\gamma + t^\gamma \log \frac{1}{t} \text{ for } 0 < t \leq 1 \\ t^{-\gamma} + t^{-\gamma} \log t \text{ for } t > 1 \end{cases} .$$

This is also a continuous version of a result of Nahmod [N1].

THEOREM 7. *There exists $\alpha_0, 0 < \alpha_0 < \gamma$, such that for $0 < \alpha < \alpha_0$ the operator T_α has a bounded inverse in L^2.*

PROOF. It follows from Theorem 6 that for $f \in \mathrm{Lip}_0(\beta)$, $0 < \beta < \gamma$

$$-\alpha^2 \, T_\alpha f = \int_0^\infty s^{-\alpha} Q_s \left(\int_0^\infty t^\alpha Q_t f \frac{dt}{t} \right) \frac{ds}{s}. \tag{8.1}$$

On the other hand it is known, see for example [C], that

$$f = \int_0^\infty Q_s \left(\int_0^\infty Q_t f \frac{dt}{t} \right) \frac{ds}{s}. \tag{8.2}$$

Using the continuous version of Coifman's and Nahmod's formula (see [C] and [N2]) (8.1) and (8.2) are respectively equal to

$$\int_0^\infty t^\alpha w_t f \frac{dt}{t}$$

and

$$\int_0^\infty w_t f \frac{dt}{t}$$

where

$$w_t f = \lim_{M\to\infty} \int_{\frac{1}{M}}^M Q_s Q_{ts} f \frac{ds}{s},$$

and the limit is in L^2_μ norm.

Using the above formulas we have

$$\| f + \alpha^2 T_\alpha f \|_2 \leq \int_0^\infty | 1 - t^\alpha | \quad \| w_t f \|_2 \frac{dt}{t},$$

where $\| \|_2$ is the L^2_μ norm.

PROOF. By Lemmas 8 and 9

$$\|w_t\| \leq c_t$$

where c_t is the constant in Lemma 9, hence

$$\int_0^\infty | 1 - t^\alpha | \| w_t f \|_2 \frac{dt}{t} \leq \int_0^\infty | 1 - t^\alpha | c_t \frac{dt}{t} \| f \|_2 .$$

To estimate the last integral write it as the sum

$$\int_0^{\frac{1}{N}} \mid 1 - t^{\alpha} \mid c_t \frac{dt}{t} + \int_{\frac{1}{N}}^{N} \mid 1 - t^{\alpha} \mid c_t \frac{dt}{t} + \int_N^{\infty} \mid 1 - t^{\alpha} \mid c_t \frac{dt}{t} =$$
$$= I_1 + I_2 + I_3.$$

Using the estimate for c_t in Lemma 9, we can find $N = N_0$ sufficiently large so that I_1 and I_3 are less than $\frac{1}{4}$ uniformly with respect α with α in $(0, \gamma']$ for a fixed γ' less than γ. Having chosen N we can find an α_0 so that for $0 < \alpha < \alpha_0$, I_2 is less than $\frac{1}{2}$. Therefore $\left\| I + \alpha^2 T_\alpha \right\| < 1$, and hence $-\alpha^2 T_\alpha$ and therefore so is T_α. □

□

Finally, in [**HV**] Hartzstein and Viviani have shown that $T_\alpha^{-1}$is also a Calderón-Zygmund operator and consequently bounded in all L^p spaces.

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