1 Overview

My primary research lies at the intersection of representation theory of associative algebras (quivers) and invariant theory. More specifically, the study of the geometry of representation spaces for string algebras.

Background. Quivers and quivers with relations appeared first in connection to representation theory of finite-dimensional associative algebras. Since their introduction, they have been used to study a range of important questions including the Deligne-Simpson problem ([8]), saturation for Littlewood-Richardson coefficients ([11]), cluster algebras ([5], [6], [12],[23]), and algebraic geometry ([7], [11], [24]). Drozd showed in [14] that there is a trichotomy in the representation types of quivers (an quivers with relations): either there are only finitely many indecomposable representations (finite type), at most finitely many one-parameter families of indecomposables in each dimension vector (tame type), or the category of modules over the free algebra on two non-commuting variables can be embedded into the category of representations (wild type). In the former two cases, it is feasible to explicitly describe all indecomposable representations.

Quivers of finite and tame representation type were classified by Gabriel ([15]) and Nazarova ([22]), respectively. Gabriel shows that the quivers of finite representation type are those whose underlying undirected graphs are simply laced Dynkin diagrams, and Nazarova shows that those of tame representation type correspond to extended Dynkin diagrams.

In the context of quivers with relations, far less is known. One family that has been shown to be of tame representation type are the string algebras, as defined in [3] and motivated by work of Gelfand-Ponomarev [17]. These algebras are interesting in that the indecomposable representations can be described as certain walks on the quiver ([4], [21], [28]), and Auslander-Reiten sequences can be constructed by performing operations on these walks. One of the exceptional aspects of these algebras is that while they are tame, the number of one-parameter families of indecomposable representations grows non-polynomially with the dimension (in contrast with the example of tame quivers without relations).

Rings of Semi-Invariants. One natural question is what data the geometry of the moduli spaces of representations captures about the category of representations. If

$\mathbb{C}Q/I$ is a quiver with relations and β is a dimension vector, we denote by $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)$ the variety of representations of $\mathbb{C}Q/I$ of dimension β. This space admits an action of $\mathrm{GL}(\beta)$, a suitable product of general linear groups, by change of basis, and orbits correspond to isomorphism classes of representations. The rings $\mathbb{C}[\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)]^{\mathrm{SL}(\beta)}$ often reflect the nature of the category of representations of the quiver with relations. The following are natural problems.

Problem 1: Characterize tame type by the geometry of representation spaces. Describe their rings of semi-invariants.

Problem 2: Determine degree bounds for the generators and relations of rings of (semi-)invariant functions.

In the context of problem 1, Skowroński-Weyman [26] studied these rings of semi-invariants for quivers (without relations). They showed that if the quiver is of finite type, the rings of semi-invariants are polynomial rings, if it is of tame type, the rings of semi-invariants are complete intersections. In wild type, however, there are dimension vectors in which the rings of semi-invariants are not complete intersections.

Following this work, a number of authors have attempted to describe the rings of semi-invariants for quivers with relations that are tame (see [1], [13], [19], [27], [29]). In particular, Kraśkiewicz ([19]) constructed examples of string algebras (so algebras of tame type) whose rings of semi-invariants are not necessarily complete intersections. This would indicate that a more thorough inspection of these algebras is required.

Semi-Invariants for Gentle String Algebras. In my first paper, I describe a procedure to construct the rings of semi-invariants for components of $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)$ when $\mathbb{C}Q/I$ is a gentle string algebra. The crucial observation in this case is that the representation spaces are products of varieties of complexes, as studied by DeConcini-Strickland in [9]. They describe the irreducible components and coordinate rings of these varieties via Schur modules. This allows us to show that the rings of semi-invariants are semi-group rings, which are in fact coordinate rings of toric varieties. These semi-groups can be encoded by so-called *matching graphs* in such a way that generators correspond to particular walks on the graph, and relations correspond to certain configurations. In this way, explicit degree bounds for the generators and relations can be determined.

Generic Representations. Another question concerning representation spaces is a problem originally posed by Kac [18]. Let Q be a quiver, and β a dimension vector. In each component of the representation space, does there exist an open (or dense) set U, and a decomposition of β into a sum of dimension vectors $\beta(1) + \ldots + \beta(s)$ such that for each representation V in U, V decomposes into a direct sum of indecomposable representations $V(1) \oplus \ldots \oplus V(s)$ where $\dim V(i) = \beta(i)$? Such a decomposition of β is referred to as its canonical decomposition, and the representations in the open set are called generic representations. In the same article, Kac answers the question in the affirmative, although the proof is non-constructive. Later, Derksen-Weyman [10] and Schofield [25] gave independent algorithms for determining this canonical decomposition in the case of representations of quivers.

Problem 3: Given a quiver with relations, and an irreducible component of $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)$, determine the canonical decomposition of β, and the generic representations (with respect to the irreducible component).

GIT Quotients. The result of Kraśkiewicz (a tame algebra with a ring of semi-invariants which is not a complete intersection) shows that geometric conditions on the rings of semi-invariants may not be fine enough to characterize tameness. Fixing a weight χ for $\mathrm{GL}(\beta)$ once can construct the GIT quotient $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)//_{\chi}\,\mathrm{GL}(\beta) := \mathrm{Proj}\bigoplus_{n\geq 0}\mathbb{C}[\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)]^{\chi^n}$. It has been shown by Chindris [7] that for path algebras of quivers, this GIT quotient is simply a projective space.

Problem 4: Characterize tameness by geometric conditions on the GIT quotient.

Generic Representations for Gentle String Algebras. In my second paper, motivated by the combinatorics involved in determining the generators for the rings of semi-invariants, and prior work by Kraśkiewicz-Weyman [20], I describe a method for constructing the generic representations in irreducible components of $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta$ when $\mathbb{C}Q/I$ is a gentle string algebra. This description allows us to determine the GIT quotient in certain cases. Namely, in the case that the generic representation of $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)$ is a band, we can indeed show that the GIT quotient is a product of projective spaces. The method of proof in this case amounts to showing that certain modules have no self-extensions, and such modules are often encountered in tilting theory, quiver Grassmannians, and cluster algebras from surfaces.

Future Research. I am exploring a number of new questions. It is well-known that string algebras are quotients of gentle string algebras, so the techniques developed thus far may be able to be applied to string algebras in general. It is a simple consequence of DeConcini-Strickland [9] that components of $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)$ are Cohen-Macaulay and normal when $\mathbb{C}Q/I$ is gentle, but it is not clear whether or when these properties hold for components of string algebras. In fact, even irreducible components of string algebras are not known.

The construction of the generic module in an irreducible component of $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)$ described above does not spell out the canonical decomposition of a given dimension vector, although it can be determined by hand on any example. It would be beneficial to determine the dimension vectors in which the generic representations are indecomposable (Schur representations), and specifically when they are indecomposable string modules. Because the proof that the constructed representations are generic involves showing the vanishing of self-extension groups, the method could be employed to determine tilting modules, which are useful in the study of cluster algebras arising from surfaces.

2 My Results

Definition 2.1. A *gentle string algebra* is an algebra with presentation $\mathbb{C}Q/I$ satisfying the following:

a. Every vertex is the source of and target for at most two arrows (respectively);

b. For every arrow $b \in Q_1$ there is at most one arrow a and at most one arrow c as in the diagram $\xrightarrow{a}\xrightarrow{b}\xrightarrow{c}$ such that $ba \notin I$, and $cb \notin I$;

c. For every arrow $b \in Q_1$ there is at most one arrow a and at most one arrow c as in the diagram $\xrightarrow{a}\xrightarrow{b}\xrightarrow{c}$ such that $ba \in I$, and $cb \in I$;

d. I is quadratic (generated by 2-paths).

A string algebra is an algebra with conditions (c) and (d) removed.

Proposition 2.2 (C)**.** *Every acyclic string algebra is a quotient of a gentle string algebra.*

Definition 2.3 (C)**.** A *coloring* of a quiver Q is a map $c : Q_1 \to S$ (S is some finite set of *colors*) satisfying $c^{-1}(s)$ is a path for each $s \in S$. The colored ideal I_c is the ideal in $\mathbb{C}Q$ generated by the set $\{ba \mid c(b) = c(a), h(a) = t(b)\}$.

Proposition 2.4 (C)**.** *Every acyclic gentle string algebra* $\mathbb{C}Q/I$ *admits a coloring* c *such that* $I = I_c$*.*

This is NOT true when Q has cycles. Henceforth let us call $\mathbb{C}Q/I$ a *colored string algebra* when it is gentle and admits a coloring.

Example 2.5. GSA-example.pdf

Remark 2.6. Suppose that $\mathbb{C}Q/I$ is a gentle string algebra, with a coloring c. Then the variety $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)$ is a product of varieties of complexes.

This is clear, for a representation consists of the assignment of a linear map to each arrow, and the ideal gives that the composition of two maps of the same color should be zero.

Definition 2.7. Let $\mathbb{C}Q/I_c$ be a colored string algebra. A *rank map* for a dimension vector β is a map $r : Q_1 \to \mathbb{N}$ such that $r(a)+r(b) \leq \beta_x$ whenever $\xrightarrow{a} x \xrightarrow{b}$ and $c(a) = c(b)$. Define the partial order $\preceq$ on the set of rank sequences for β by $r \preceq r'$ if and only if $r(a) \leq r'(a)$ for each $a \in Q_1$.

Proposition 2.8 (Independent proof Carroll)**.** *(Corollary to DeConcini-Strickland "On Varieties of Complexes") The irreducible components of* $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)$ *are given by*

$$\mathrm{Rep}_{\mathbb{C}Q/I}(\beta, r) := \{M \in \mathrm{Rep}_{\mathbb{C}Q/I}(\beta) \mid \mathrm{rank}(M_a) \leq r(a)\}$$

for r *maximal.*

There is a filtration on the coordinate ring of each variety of complexes whose associated graded is given by a nice formula. This extends to $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta, r)$, and the result is given below, after a few preliminary notations:

a. Let $\Lambda_r = \{\lambda : Q_1 \to \mathcal{P} \mid \lambda(a)$ has at most $r(a)$ parts$\}$, where $\mathcal{P}$ is the set of partitions;

b. Let $\mathfrak{X} = \{(x, s) \in Q_0 \times S \mid$ there is an arrow of color s incident to the vertex $x\}$;

c. Suppose $\lambda \in \Lambda_r$, and $(x, s) \in \mathcal{X}$. Let $a, b \in Q_1$ be the arrows with $c(a) = c(b) = s$ and $h(a) = t(b) = x$. Denote by $\lambda(x, s) = (\lambda(b), 0, \dots, 0, -\lambda(a))$ where $-\lambda(a) = (-\lambda(a)_{r(a)}, -\lambda(a)_{r(a)-1}, \dots, -\lambda(a)_2, -\lambda(a)_1)$, and $\lambda(x, s)$ is of length β_x (i.e., we put enough zeros to fill a vector of length β_x);

d. If μ is a non-increasing sequence of integers of length n, then $S_\mu\mathbb{C}^n$ is the corresponding Schur module. That is $S_{(1,1,1,\dots,1,0,0,\dots,0)}\mathbb{C}^n = \bigwedge^l \mathbb{C}^n$ and $S_{(l,0,\dots,0)}\mathbb{C}^n = S^l\mathbb{C}^n$.

Proposition 2.9. *Let $\mathbb{C}Q/I$ be a colored string algebra, supplied with a coloring $c : Q_1 \to S$. Let β be a dimension vector, and r a maximal rank sequence. Then*

$$\mathbb{C}[\mathrm{Rep}_{\mathbb{C}Q/I}(\beta, r)] = \bigoplus_{\lambda \in \Lambda_r} \bigotimes_{(x,s) \in \mathfrak{X}} S_{\lambda(x,s)} V_x$$

While this is true as stated, the result is slightly deeper: there is a filtration on $\mathbb{C}[\mathrm{Rep}_{\mathbb{C}Q/I}(\beta, r)]$ whose associated graded algebra is that given above. This result holds in more generality, as there are many zero-relation algebras which can be colored. The combinatorics in the case of gentle string algebras are quite nice, since at each vertex we have the tensor product of at most two Schur modules.

Definition 2.10. For each $x \in Q_0$, let $s_1(x), s_2(x)$ be the elements of S such that $(x, s_1(x)), (x, s_2(x)) \in \mathfrak{X}$. Let Λ_r^{SI} be the set of $\lambda \in \Lambda_r$ such that for each $x \in Q_0$, $\lambda(x, s_1(x))_i - \lambda(x, s_1(x))_{i+1} = \lambda(x, s_2(x))_{\beta_x - i} - \lambda(x, s_2(x))_{\beta_x - i+1}$. If $s(x)$ is the unique color incident to $x \in Q_0$, then the requirement reads $\lambda(x, s(x))_i - \lambda(x, s(x))_{i+1} = 0$.

Theorem 2.11 (C)**.** *Let $\mathbb{C}Q/I$ be a gentle string algebra with coloring c, dimension vector β, and maximal rank sequence r.*

a. Λ_r^{SI} is a semi-group with respect to component-wise addition of vectors, and $\mathbb{C}[\Lambda_r^{SI}]$ is a polynomial ring over a semi-group ring $(\Lambda_r^{SI})^\circ$;

b. $\mathbb{C}[\mathrm{Rep}_{\mathbb{C}Q/I}(\beta, r)]^{\prod_{x \in Q_0} \mathrm{SL}(\beta_x)} \cong \mathbb{C}[\Lambda_r^{SI}]$, where the latter is the semigroup ring;

c. $\mathbb{C}[\Lambda_r^{SI}]$ *is the coordinate ring of a toric variety.*

The combinatorics of the proof for part (a) are interesting, but they lead most notably to the following:

Theorem 2.12 (C)**.** *Same assumptions as before. The ring* $\mathbb{C}[\mathrm{Rep}_{\mathbb{C}Q/I}(\beta, r)]^{\mathrm{SL}(\beta)}$ *is generated by elements of degree bounded by*

$$\sum_{a \in Q_1} 2\binom{r(a)+1}{2}$$

The proof of this theorem relies on an explicit description of generators and relations for the semi-group $(\Lambda_r^{SI})^{\circ}$. This semi-group is an example of a so-called *matching semigroup*, which I define in the first paper.

Definition 2.13. Let $f_1, \dots, f_{2m}$ be a collection of $\mathbb{N}$-linear functionals $f_i : \mathbb{N}^l \to \mathbb{N}$, written $f_i(\underline{x}) = \sum_{j=1}^{l} \alpha_i^j x_j$. Let $U(\underline{f})$ be the set $\{u \in \mathbb{N}^l \mid f_i(u) = f_{i+m}(u)$ for all $i = 1, \dots, m\}$. The semi-group $U(\underline{f})$ is called a *matching semigroup* if

- $\alpha_i^j \in \{0, 1\}$ for $1 \le i \le 2m$, $1 \le j \le l$;
- for each i, the set $\{j \in [1, 2m] \mid \alpha_i^j \neq 0\}$ has at most two elements.

Simply put, matching semigroups are given by equalities of certain $\mathbb{N}$ linear functions, and in the set of defining equations, each variable shows up at most twice. This latter point is quite important, because it allows us to build a graph from the defining equations, and edges can correspond to variables.

Definition 2.14. Suppose that $U(\underline{f})$ is a matching semigroup, defined by $2m$ functionals. Define the graph $G(\underline{f})$ as follows: $G(\underline{f})$ has $2m$ vertices, one corresponding to each functional f_i, and edges of the following type:

- a dotted edge $i - - - i+m$ for $i = 1, \dots, 2m$;
- a solid edge labeled x_t $\; i \overset{x_t}{\text{———}} j$ if $\alpha_i^t = \alpha_j^t = 1$;

- a solid self-loop labeled x_t at i if $\alpha_i^t = 1$ and is the unique such.

The crucial point is certain walks on the graph $G(\underline{f})$ correspond to elements of $U(\underline{f})$, and in fact that every element of $U(\underline{f})$ can be realized as a walk on the graph. Namely:

Definition 2.15. A walk w on the graph $G(\underline{f})$ is called an *alternating string* if the first and last edge are loops, and the edge types alternate (solid-dotted-solid-...); and is called an *alternating band* if w is a cycle, the first edge is dotted, the last edge is solid, and the edge types alternate. If w is a walk on $G(\underline{f})$, then define by $u(w)_j$ the number of times that the edge labeled x_j is traversed in the walk w.

Proposition 2.16 (C)**.**

- *If w is an alternating string or band, then $u(w) \in U(\underline{f})$.*
- *If $u \in U(\underline{f})$, then u can be written as a sum of $u(w)$ over alternating strings and bands. That is $\{u(w) \mid w$ is an alternating string or band$\}$ generates $U(\underline{f})$ as a semigroup (but not necessarily minimally).*
- *If $u(w)$ is irreducible (i.e., cannot be written as an $\mathbb{N}$ linear combination of others, then $f_j(u(w)) \leq 2$ for each j. In particular, $u(w)_i \leq 2$ for each i.*

The second portion of my research deals with the determination of the generic modules in irreducible components $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta, r)$. That is, determine a module (or at worst finite collection of one-parameter families of modules) in $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta, r)$ so that the orbit under $\mathrm{GL}(\beta)$ (or the union of the $\mathrm{GL}(\beta)$ orbits) is dense.

The reason for this is the following: there is a second description of rings of semi-invariants as follows. Let $\mathbb{C}Q/I$ be an arbitrary quiver with relations, β a dimension vector, and $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)_z$ an irreducible component of the representation space. There is a bilinear form on the Grothendieck group of the category $\mathrm{Rep}_{\mathbb{C}Q/I}$ given by

$$< \dim V, \dim W >= \sum_{i \geq 0} \dim \mathrm{Ext}^i_{\mathbb{C}Q/I}(V, W)$$

(It can be shown that this is independent of the choice of V, W in given dimension vector). If the projective dimension of V is 1,

$$V \longleftarrow P_0(V) \xleftarrow{\delta_0} P_1(V) \longleftarrow 0$$

we can apply $\mathrm{Hom}_{\mathbb{C}Q/I}(-,W)$ to this resolution,

$$0 \to \mathrm{Hom}_{\mathbb{C}Q/I}(V,W) \to \mathrm{Hom}_{\mathbb{C}Q/I}(P_0(V),W) \xrightarrow{d_W^V} \mathrm{Hom}_{\mathbb{C}Q/I}(P_1(V),W) \to \mathrm{Ext}^1(V,W) \to 0$$

is an exact sequence. If $< V, W >= 0$, then (after identifying basis) d_W^V is a square matrix. Derksen-Weyman prove the following:

$$\mathbb{C}[\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)_z]^{\mathrm{SL}(\beta)} = \{c^V(-) = \det(d_-^V \mid p.d.(V) \leq 1, \ < \dim V, \dim W >= 0\}$$

In addition, the weight of $c^V(-)$ is $< V, - >$. So in order to describe $\mathbb{C}[\mathrm{Rep}_{\mathbb{C}Q/I}(\beta)_z]^{\mathrm{SL}(\beta)}$, it is enough to determine those V in the category $\beta^\perp = \{V \in \mathrm{Rep}_{\mathbb{C}Q/I}(\beta) \mid\}$

The impetus for determining the general elements in the irreducible components is then clear: the weight spaces will be spanned by c^V for V general. Motivated by the combinatorics involved with calculating semi-invariants, we give an explicit construction of a dense subset in $\mathrm{Rep}_{\mathbb{C}Q/I}(\beta, r)$ when $\mathbb{C}Q/I$ is a gentle string algebra. This consists of defining a module (or family of modules) so that the orbit (or the union of the orbits) is dense. In the case that the modules that we construct are rigid (admit no self-extensions), the result follows from Voigt. [**?**]. When there is a family of modules, the result also relies on showing that the general extension groups are trivial.

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