Report on the Use of HK4 in a Class for Teachers

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The DePaul University course called Discrete Structures for Teachers is part of the Master of Arts in Mathematics Education program. The majority of students in the program currently work as high school or middle school teachers. The rest are students whose undergraduate degrees are in mathematics-related areas, who worked for a while outside an educational environment, and who now want to become teachers of mathematics.

Typically students take Discrete Structures for Teachers as one of their first courses in the program. A theme of the course is an introduction to logical reasoning and proof. The text is *Discrete Mathematics with Applications*, 3rd edition, which I wrote.

Students get quite a bit of practice interpreting and negating quantified statements, and the introduction to proof emphasizes the role of what I call "generalizing from the generic particular." This is the logical principle that allows us to prove, for example, that the square of any even integer is even in the following way: We suppose we have a particular but arbitrarily chosen (or generic) even integer, say n (to give it a name), but we do not allow ourselves to assume any properties about n except for those that it shares with every other even integer. We then show that n^2 is even.[1]

Because I'm teaching teachers, I try to make certain points that are related to the material of the book but that are especially relevant to the high school curriculum. One such point is that the graph of a (real-valued) function (of a real variable) is defined to be the set of all ordered pairs for which the second coordinate is the value of the function at the first coordinate. I comment (with a smile) that I have long thought that understanding this fact is the secret of success in calculus. But a general definition of graph, such as this, is often omitted in high school textbooks. I once spoke to an author of such a textbook who stated that because students have been led through the experience of constructing graphs by plotting points in pre-algebra and beginning algebra they know what graphs are. Experience teaching calculus leads me to disagree. In my experience because students are ignorant of the general definition, they often cannot follow teacher's explanations that involve a general function f and do not understand that, for example, $f(x + h)$ is the height of the graph at $x + h$.

Another point I try to make concerns the logic of solving equations. I often begin by paraphrasing Jean Dieudonné (Abstraction in Mathematics and the Evolution of Algebra, in *Learning and the Nature of Mathematics*, W. J. Lamon, ed., 1972): "Direct analysis of an equation, strictly by algebraic methods, consists, as we know, of performing a series of operations on the unknown (or unknowns) *as if it were a known quantity*...A modern mathematician is so used to this kind of reasoning that his boldness is now barely perceptible to him."

I like this quote because of the respect it implies for those who are new to the use of this technique. To give an unknown quantity a name and then operate on it as if one knew what it was *is* truly bold. Even mathematically advanced students are often too timid to do it in unfamiliar situations. Teachers should be proud of themselves when they succeed in getting their students to start using it in at least a few settings.

Also I caution teachers against too early and too exclusive a use of the phrase "Solve the equation" because I strongly suspect that although we (mathematicians) may well understand these words to

[1] To fill in the reasoning: By definition of even, $n = 2k$ for some integer k, and so $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is twice some integer and hence even by definition of even.

mean “Find all values of the variable(s) that make the equation true,” some students come to see them as simply initiating a sequence of formal steps involving the mysterious “*x*” to obtain an answer that makes their teacher happy. So I urge students to go back and forth between the formal “Solve” language and the more descriptive “Find all values of” language in order to keep reinforcing the meaning of “solve that” in students’ minds.

I follow up by discussing the thought processes involved in the steps of solving an equation. If one is given an equation, say in x, to solve, one’s job is to find all values of x for which the equation is true, i.e., for which the left-hand side equals the right-hand side. Many beginning algebra books talk about “equivalent equations” and the operations that preserve the “solution set.” My impression is that this approach is too abstract for most students to understand fully. Moreover, it is not generally reinforced later in the books.

Indeed, later on in many students’ experience of algebra, the reasons given to justify going from one step to another in the solution process convey a different impression entirely. For instance, to solve $2x – 1 = 7$, the reasons given might be (1) add 1 to both sides, and (2) divide both sides by 2. These reasons clearly indicate why if x is a number for which $2x – 1 = 7$, then (1) $2x = 8$ will also be true, and thus (2) $x = 4$ is also true (because if two numbers are equal and one adds 1 to both or divides both by 2 the results will also be equal). But the reasons given for the steps do not at all justify the reverse implication: that if $x = 4$, then $2x – 1 = 7$. Thus when students encounter equations with no solutions, quadratic equations with redundant solutions, and identities, they may understandably be confused and end up writing “correct answers” through a purely rote process. (“If I do these operations and come up with this result, I’m supposed to answer in such-and-such a way.”)

When I taught the course in the fall of 2004, I gave students a worksheet with the following directions and problems:

> Find all real numbers x for which the following equations are true. Write your answers in a way that makes your reasoning clear.
>
> **(a)** $\frac{1}{12-4x} = \frac{1}{x^2-5x+6}$
>
> **(b)** $(2x + 1)(x + 1) + 4 – x = (x + 2)(2x – 2)$
>
> **(c)** $5(x – 3) – 3(x – 1) = 2(x – 6)$

When one applies the usual methods for solving equations, one obtains the following results: If x is a number that satisfies equation (a), then $x = –2$ or $x = 3$. But when one checks, one finds that only $x = –2$ satisfies the equation. If x is a number that satisfies equation (b), then $5 = –4$, which is false. So there is no number that satisfies equation (b). And if x is a number that satisfies equation (c) then $0 = 0$.

Students did the worksheet in class, and then we discussed it. In the discussion I tried to make clear how the examples illustrated various aspects of the logic of equation solving. The main point I made is that we typically think of solving equations as an if-then process, not an if-and-only-if process. And for that reason, we need to check answers by plugging into the left-hand side (LHS) and the right-hand side (RHS) of the original equation to see if they are equal. (See Endnote 1 for more discussion of why I emphasize LHS and RHS.)

I then gave a similar sheet for students to do as homework (Appendix 1). I followed up in the next class by going over solutions for each of the homework exercises, which I also distributed (Appendix 2). Then I showed sections of the TIMSS video HK4: 00:34 – 08:48, 11:18 – 19:13, and 27:34 – 31:53.

Following the video, I talked about what impressed me about the lesson and the students interjected and added their comments for a lively discussion that probably lasted 15-20 minutes. Afterwards I asked if they found the video worthwhile, and they assured me with considerable enthusiasm that they did.

Here are my notes about the discussion. Unfortunately most are things that I said that I had jotted down beforehand, but others are elaborations and additions suggested by the students that I wrote down at the time or remember now. I wish I had thought to write down everything the students said.

(1) I said that I chose the lesson to show to this class because it lays the groundwork – for eighth graders(!) – for so many of the ideas that were themes of our course and are basic to more advanced mathematical thinking. My students added some further examples I hadn't noticed.

- What it really means to solve an equation and what it means for an equation to be true – find values of the variable(s) for which LHS = RHS.
- Understanding of quantification (that some equations are true for *some* values of the variable whereas others are true for *all* values of the variable, also practice in expressing what it means for a quantified statement to be false) (00:19:13).
- Introduction to the idea of determining whether an equation is an identity at a very early stage of the curriculum, using quite simple equations. Often in the U.S. the term identity (meaning equation that is true for all possible values of its variable(s)) is only used early on in connection with properties of real numbers such as the "commutative identity" etc., but the meaning of the word itself is not explained. When students encounter it later in connection with trigonometric identities, the focus is on computational techniques rather than the meaning of the word.
- Introduction to the idea of generalizing from the generic particular – the logical principle that if one can establish a property for a particular but arbitrarily chosen element of a set, then the property is true for all elements of the set. (This is the principle the Hong Kong teacher finally has the students use to establish that equations are identities.)
- Making clear the flow of a mathematical argument through frequent use of words like "therefore." (A student commented on this.)
- The idea of disproving a universal (general) statement with a counterexample. (00:18:35)
- The related idea that a few examples do not suffice to prove a general statement. (Another student was struck by the fact that the teacher not only said that it was not sufficient to show the equation was true for 5 values, but actually went on to clarify what this meant by noting that the equation might not be true for a sixth value.)
- Helping students develop an understanding of the use of variables in universal statements through the use of empty boxes to denote "general objects" when discussing the meaning of expanded form and factorized form. (A student commented on this. I had used empty boxes in a similar way when discussing, for instance, the definition of composition of functions.)

(2) I also said that I admired the way the lesson started, with students working two contrasting examples, both of which made use of skills they had learned previously but which had an unexpected outcome that led to a question to catch students' interest.

(3) It seemed to me that the slow and deliberate pace of the 32-minute lesson seemed to be designed to enable as many students as possible to build understanding and develop skill and yet maintain the interest of the better students. This appealed to my hope that techniques can be found to enable a larger group of students to be successful with algebra.

- When a few of the Hong Kong students indicated that they understood that the result obtained for the second equation indicated that it was true for any number, I realized that had I been the teacher I might well have immediately said "Yes!" and launched into the definition of identity, etc. Instead the teacher had the students spend the next 6 or 7 minutes checking the left- and right-hand sides of the equations, first all together and then individually. This gave his students practice in making substitutions but in a way that might not turn off the better students because the computations were in the service of exploring a question they might find interesting. The process also reinforced for students the meaning of what it means for an equation to be true because they had to check both the LHS and the RHS and see whether they were equal. And taking the time to do the computations presumably gave the slower students an opportunity to become more comfortable with the ideas.
- Many times the teacher asked his students "What does this mean?" Even though he often appeared to answer the question himself, simply asking it communicated the importance of understanding what things mean in mathematics.
- My students were impressed by the emphasis the teacher placed on the word "identity" and the phrase "identically equal." I also thought this sent a useful message about the importance of using terminology correctly.
- My students also praised the Hong Kong teacher's insistence at the end of the period that his students not just answer "yes" or "no" but write down whether the left- and right-hand sides were equal and state their conclusion.

One student commented that if a supervisor came in for only a few minutes of the lesson during which the teacher was lecturing, the supervisor might downgrade the teacher for not having the class work in groups. That was a discouraging comment.

A requirement of my course was that students develop a lesson or lesson segment based on something they learned in the course. One student, Tanya DeGroot, based her lesson segment on the work we had done of the logic of equation solving and the meaning of graph of an equation, and the insights she had gained from viewing HK4. (See Appendix 3 for her report and Appendix 4 for the full worksheet she prepared for her class.) Tanya was a good student who well deserved the A she earned in the class. One comment in her report jumped out at me. I've given its context and put it in italics: "As in the Hong Kong lesson, I asked students to think of each equation as the left-hand side and the right-hand side. The equation was only true if the left-hand side and right-hand side were equal to each other. *I had not seen this approach prior to our work in Discrete Mathematics*..." When she describes her students' reaction to her lesson, she also makes clear that this idea was new to all of them too – so new that some of them had a hard time grasping it. And these were students who had had a year of algebra and a year of geometry.

Endnote 1: When students in my class write proofs of formulas by mathematical induction, in both the basis and the inductive steps, they often write the equation they want to be true (thereby effectively assuming it) and then deduce something they know to be true (such as 1 = 1). From a logical point of view this is not a proof at all, even though the steps can be turned into a proof if rewritten appropriately. In the three editions my book has gone through, I have used increasingly explicit measures to counteract this problem. For instance, in the proof for the formula for the sum of the first n integers, in the first edition I simply wrote, "We must show that $1 + 2 + 3 + \cdots + (k + 1) = (k + 1)(k + 2)/2$. But $1 + 2 + 3 + \cdots + (k + 1) = \ldots\ldots\ldots\ldots = (k + 1)(k + 2)/2$." When I taught from the book, it became clear that many students did not understand the logic of the calculations, and so, in the second edition, after the "We must show" sentence and before the "But," I added "*[We will show that the left-hand side of this equation equals the right-hand side.]*" Yet even this clarification was not sufficient for some students. So in the third edition I number the equation to be shown, and after the "But" I now write: "the left-hand side of equation (xx) is ...." and after I've shown it equal to the right-hand side, I write, “which is equal to the right-hand side of equation (xx).” Or in some problems, I simplify the left-hand side and the right-hand side separately and show that they are equal to each other. (You’ll see later in this report why I’ve given this account in such detail. For students who still resist using this method, I offer an alternative, logically correct way of writing the proofs using if-and-only-if explicitly.)

Endnote 2: I wish that the teacher in HK4 had used a different term for equations that are not identities. In fact, I would have shown a little more of the video, but I didn’t want to show the part where he distinguishes “identities” and “equations” as if they belong to disjoint sets. Also it seemed to me that it is not a good idea to use the not-equal sign to denote an equation that is not an identity because the two sides of the equation may be equal for certain values of the variable. I wish he had used the not-identically-equal sign (a slash through three parallel lines).

NAME: ____

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Appendix 1 – The Logic of Equation Solving

Express the logic of your solutions clearly as you do the problems below.

1. Solve the equation: $(3 - 2x)(x + 1) = x + 7 - 2(x^2 - 2)$.

2. Solve the equation: $4 - x = \sqrt{x-2}$

3. Solve for x: $x(x + 7) = (x + 2)^2 + 3x - 4$.

Appendix 2 – The Logic of Equation Solving: Solutions

Express the logic of your solutions clearly as you do the problems below.

1. Solve the equation: $(3 - 2x)(x + 1) = x + 7 - 2(x^2 - 2)$.

Solution: Suppose x is a real number for which

$$(3 - 2x)(x + 1) = (x + 7 - 2(x^2 - 2)$$

$\Rightarrow \quad 3x + 3 - 2x^2 - 2x = x + 7 - 2x^2 + 4$ by multiplying out

$\Rightarrow \quad -2x^2 + x + 3 = x - 2x^2 + 11$ by combining like terms

$\Rightarrow \quad 3 = 11$ by adding $2x^2 - x$ to both sides

But it is false that 3 = 11. Therefore, the given equation has no solution.

2. Solve the equation: $4 - x = \sqrt{x-2}$

Solution: Suppose x is a real number for which

$$4 - x = \sqrt{x-2}$$

$\Rightarrow \quad 16 - 8x + x^2 = x - 2$ by squaring both sides

$\Rightarrow \quad x^2 - 9x + 18 = 0$ by subtracting x and adding 2 to both sides

$\Rightarrow \quad (x - 3)(x - 6) = 0$ by factoring

$\Rightarrow \quad x = 3$ or $x = 6$ by the zero product property.

So the only possible solutions are $x = 3$ and $x = 6$. But when $x = 6$, the LHS of the equation is $4 - x = 4 - 6 = -2$, and the RHS is $\sqrt{x-2} = \sqrt{6-2} = \sqrt{4} = 2$. Thus, when $x = 6$, the RHS of the equation does not equal the LHS, and so 6 is not a solution. On the other hand, when $x = 3$, the LHS of the equation is $4 - x = 4 - 3 = 1$, and the RHS is $\sqrt{x-2} = \sqrt{3-2} = \sqrt{1} = 1$ also. Thus 3 is the only solution to the equation.

3. Solve for x: $x(x + 7) = (x + 2)^2 + 3x - 4$.

Solution: Suppose x is a real number for which $x(x + 7) = (x + 2)^2 + 3x - 4$

$\Rightarrow \quad x^2 + 7x = x^2 + 2x + 4 + 3x - 4$ by multiplying out

$\Rightarrow \quad x^2 + 7x = x^2 + 7x$ by combining like terms

$\Rightarrow \quad 0 = 0$ by subtracting $x^2 + 7x$ from both sides

Does this result mean that the original equation is true for all real number x? Let's check. For all real numbers x,

$$\text{LHS} = x(x + 7) = x^2 + 7x, \quad \text{and}$$

$$\text{RHS} = (x + 2)^2 + 3x - 4 = x^2 + 2x + 4 + 3x - 4 = x^2 + 7x\,.$$

So LHS = RHS. But equals equal to equals are equal to each other. Thus $x(x + 7) = (x + 2)^2 + 3x - 4$ no matter what real number is substituted for x.

Note: An equation that is true for all real numbers x is called an *identity* in x. In a Hong Kong curriculum, 8th grade students are introduced to the concept of identity by comparing equations like this one to equations with a single solution.

Appendix 3

Tanya DeGroot
Discrete Mathematics for Teachers
Dr. Susanna Epp
Lesson Segment Assignment

Logic of Representing Equations in the Cartesian Plane

Rationale and Goal for Intermediate and College Algebra

I have designed this lesson segment to use in my Intermediate and College Algebra course. The course follows study of Algebra and Geometry and includes students who are sophomores, juniors and a few seniors. These students have taken Algebra either in 8^{th} grade or freshman year and geometry in the following school year. The students are on pace with their peers at this high school, not the most advanced, but not the least advanced in mathematical study.

One challenge for me as a teacher of this course has been the tendency of students to attempt to memorize rules for each topic we study. This does not build deep understanding and makes generalizing to a new topic very difficult. Recently it became apparent that students did not have an understanding of the relationship between graphs in the Cartesian plane and equations in two variables. I have tried to make some connections more apparent and decided that this project would be a perfect opportunity to connect logic, equations and graphs to try to build deeper understandings for these students.

Description of the Lesson Segment[2]

The focus is the connection between equations in two variables and numerical and graphical representations. Students are asked to come up with ordered pairs, which make $y = x + 6$ true. As in the Hong Kong lesson, we will look at the logic (i.e. RHS = LHS) to

[2] A copy of the worksheet for students is included on pages 10-12.

determine which points should be on the graph. After students come up with several numerical solutions, we will plot the points on a graph. After plotting the points on the graph, I anticipate that students will suggest we draw in the line. We'll discuss *why* draw in the line. What does it represent? [The set of all points that satisfy the equation.] Next students will be asked to find some other solutions to the equation. Hopefully students will begin to read the points from the graph and connect that the solutions are points and points are solutions.

Students will then move on to $y = -2x + 3$ and complete the same process to find several solutions to the equation and then draw a graph. Next students will work with a linear equation in standard form: $2x + 3y = 12$. After working with these 3 linear equations we will move on to quadratic equations. Students will again be asked to find ordered pairs that make the equation true. I will guide them in finding some on each half of the parabola.

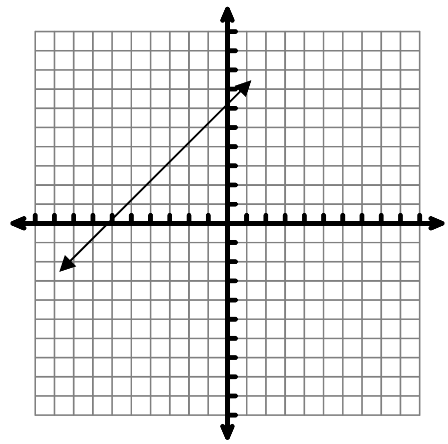
The lesson concludes with students being asked to graph several equations in two variables without the help of a graphing calculator. One equation is linear, another is quadratic and a third is rational. Students have not studied rational equations yet in this course. I hope to gain some insight into their understanding by analyzing each of these three exercises.

Relationship to work in Discrete Mathematics

In Discrete Mathematics, we discussed the "logic of equation solving" and also the Hong Kong lesson which put the logic of equation solving into practice. As in the Hong Kong lesson, I asked students to think of each equation as the left-hand side and the right-hand side. The equation was only true if the left-hand side and right-hand side were equal to each other. I had not seen this approach prior to our work in Discrete Mathematics and I believe that it can be one way to build deeper understanding of equations and also graphs for students.

Name ___________________________________ Date ____________ Period _________

1. For the equation $y = x + 6$, what are some ordered pairs $(x,\ y)$ that make the equation true?



[The typed answers and drawings on the graphs are a copy of what Tanya DeGroot included in her report]

(-6,0) (3,9) (4,10) (0,6)

Show *why* each ordered pair makes the equation true.

LHS = 0	LHS = 9	LHS = 10
RHS = -6+6	RHS = 3+6	RHS = 4+6
= 0	= 9	=10
LHS = RHS	LHS = RHS	LHS = RHS

LHS = 6
RHS = 0+6
=6
LHS = RHS

Plot these ordered pairs as points on the graph to the right.

Now find some ordered pairs that make the equation false.

(0,0) (3,4)

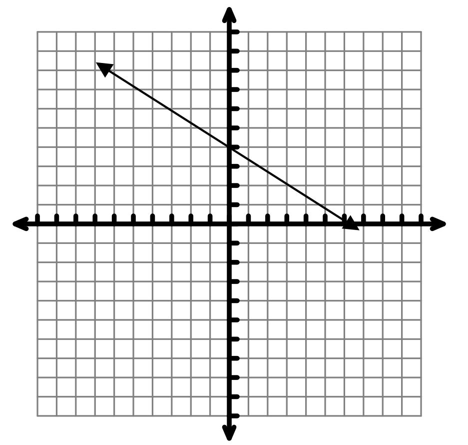
Show *why* each ordered pair makes the equation false.

LHS = 0	LHS = 3
RHS = 0+6	RHS = 3+6
= 6	= 9
LHS ≠ RHS	LHS ≠ RHS

How could you find more ordered pairs that satisfy the equation?

- pick points on your line
- guess & check

2. For the equation $2x + 3y = 12$, what are some ordered pairs $(x,\ y)$ that make the equation true?



(6,0) (3,2) (0,4)

Show *why* each ordered pair makes the equation true.

LHS = 2(6) + 3(0)
= 12
RHS = 12
RHS = LHS

LHS = 2(3) + 3(2)
= 12
RHS = 12
LHS = RHS

LHS = 2(0) + 3(4)
= 12
RHS = 12
LHS = RHS

Plot these ordered pairs as points on the graph to the right.

Now find some ordered pairs that make the equation false.

(0,0) (4,9) (1,1)

Show *why* each ordered pair makes the equation false.

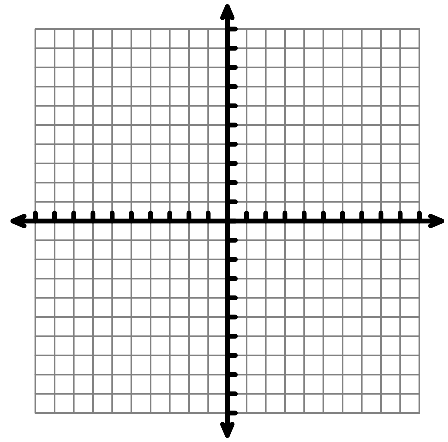
LHS = 0
RHS = 12
RHS ≠ LHS

LHS = 2(4) + 3(9)
= 35
RHS = 12
LHS ≠ RHS

LHS = 2(1) + 3(1)
= 5
RHS = 12
LHS ≠ RHS

How could you find more ordered pairs that satisfy the equation?

3. For the equation $y = x^2 - 5x + 6$, what are some ordered pairs (x, y) that make the equation true?



Show *why* each ordered pair makes the equation true.

Plot these ordered pairs as points on the graph to the right.

Now find some ordered pairs that make the equation false.

Show *why* each ordered pair makes the equation false.

How could you find more ordered pairs that satisfy the equation?

Results for this Course

At first, looking at the equations in this was difficult for many students. They did not see the necessity of looking at the two sides of the equation separately. I believe that this is because the students had never seen this approach in the past. Slowly, more students saw the pattern of what needed to be true for an ordered pair to work. In both of the classes when I used this approach a few students found ordered pairs by plugging in for x and then using the right-hand side of the equation to solve for y. The other students were amazed. I kept hearing "How is she getting all of the points correctly?" and other similar comments. I asked one student to explain how she was picking the ordered pairs that would work. She explained the process of plugging in for x and solving for y.

As the lesson continued, several students suggested ordered pairs that were not on the graph. This was a great opportunity to discuss why the ordered pairs were not on the graph, because the equation was false if the values were substituted for x and y. Some students suggested that we could find an ordered pair that would be on the graph using one that was not on the graph. For example, a student suggested that we use the ordered pair (–2,3) for the equation $y = -2x + 3$. Using this ordered pair, the LHS = 3 and the RHS = –2(2) + 3 = –4 +3 = –1. So the LHS ≠ RHS and the ordered pair should not be on this graph. But, another student observed that the ordered pair (-2,-1) should be on the graph because then the LHS = -1 and the RHS = -1 so the LHS = RHS.

Students in my second class had difficulty when we started to graph the parabola. They had found two ordered pairs that satisfied the equation. Then one student suggested another ordered pair that would have worked if the equation had been linear. But the ordered pair did not satisfy the equation. This was a mystery to several students in the class. Only after finding the ordered pair that would satisfy the equation with the same *x*-coordinate did the students begin to see that the equation would produce a parabola instead of a line.

At the end of the class period, I felt as though students had begun to connect the idea of the equation and the graph on a deeper level. However, I found that the following day, a student asked how she could graph a parabola. I replied that we could test in ordered pairs with the equation, or use an *x*-coordinate to find a *y*-coordinate and she had not thought to do that. I will continue to try to connect equations and graphs in this way and hope that more students can see the connection as the year continues.